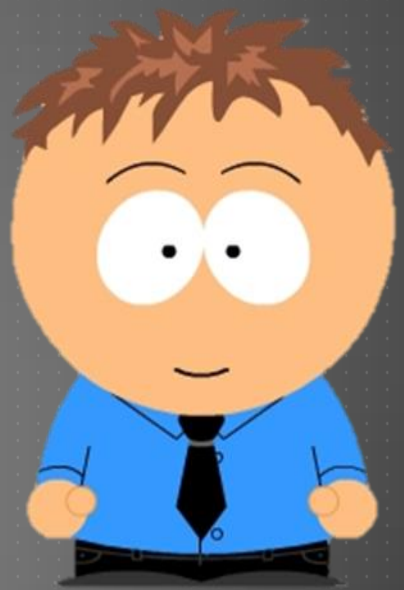
DIFFERENTIATION AND INTEGRATION PART 3

Mr C's IB Standard Notes

In this PDF you can find the following:

1. The Product Rule
2. The Quotient Rule
3. Integration by Substitution
4. Worked Examples

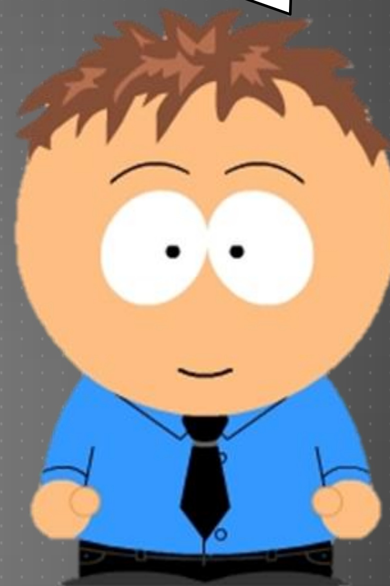
Make sure you read through everything and the try examples for yourself before looking at the solutions



THE PRODUCT RULE

Differentiating when one function is multiplied by another

Now we've seen functions like x^5 and $(x + 2)^5$, we move on to two further examples. Each has their own rule for differentiation



THE PRODUCT RULE

Differentiating when one function is multiplied by another

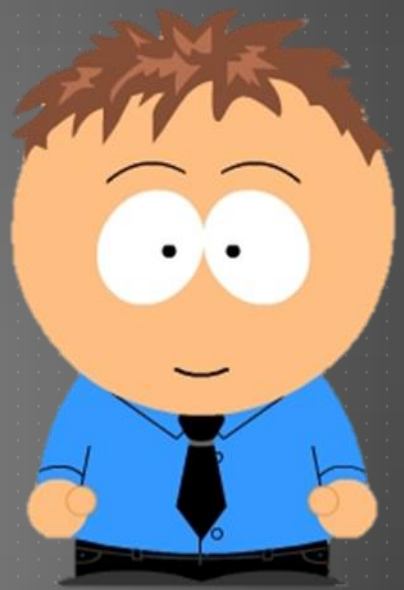
We start with functions that have been multiplied together, e.g.

$$f(x) = e^{3x}\sin(x)$$

THE PRODUCT RULE

<u>Differentiating when one function is multiplied by another</u>

$f(x) = e^{3x}\sin(x)$



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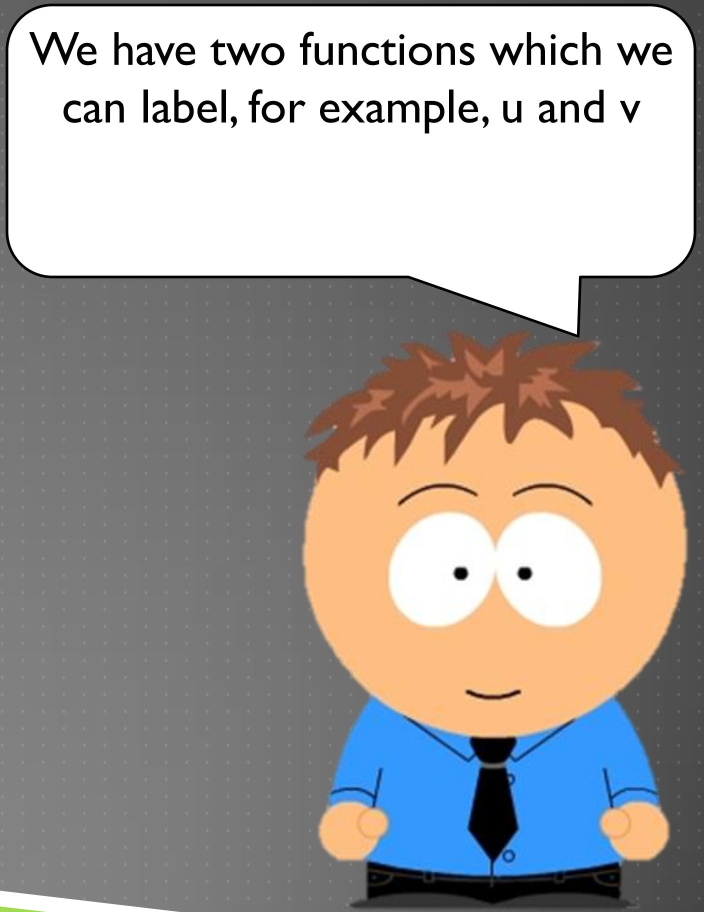
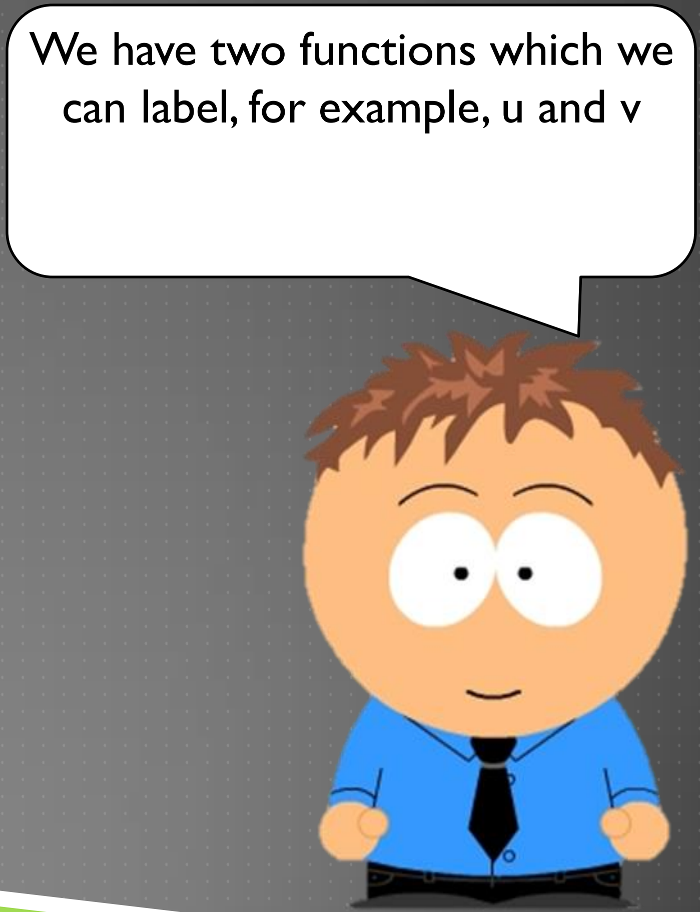


THE PRODUCT RULE

Differentiating when one function is multiplied by another

$f(x) = e^{3x}\sin(x)$

$u = e^{3x}$

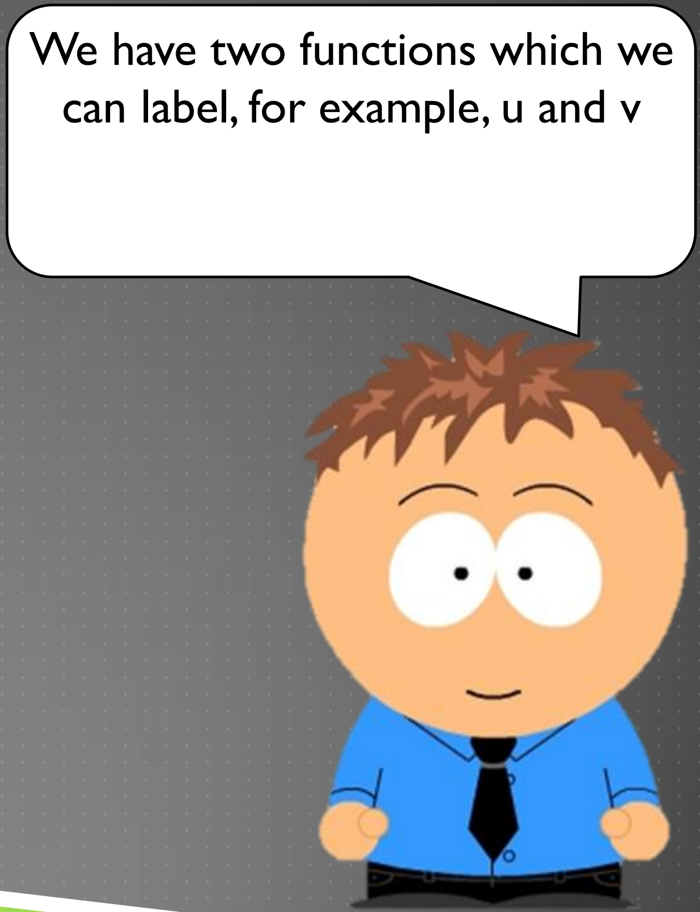


THE PRODUCT RULE

Differentiating when one function is multiplied by another

$f(x) = e^{3x}\sin(x)$

$u = e^{3x}$ $v = \sin(x)$



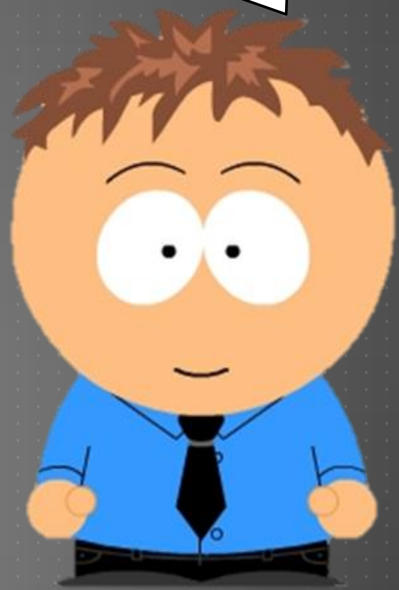
THE PRODUCT RULE

<u>Differentiating when one function is multiplied by another</u>

$f(x) = e^{3x}\sin(x)$

$u = e^{3x}$ $\qquad$ $v = \sin(x)$

The formula book tells us how to differentiate the product of u x v using method is called the *product rule*



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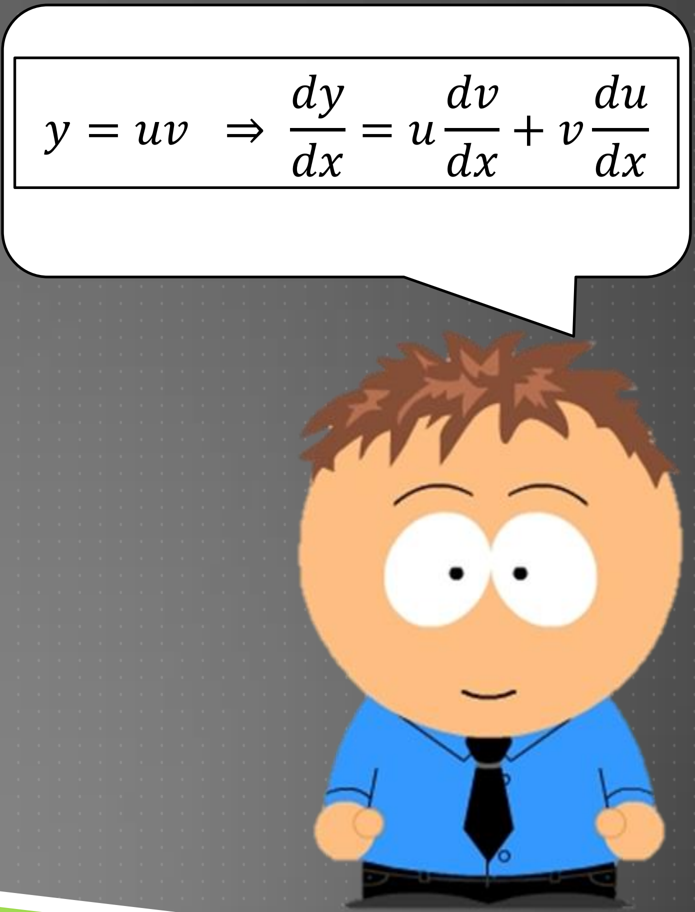

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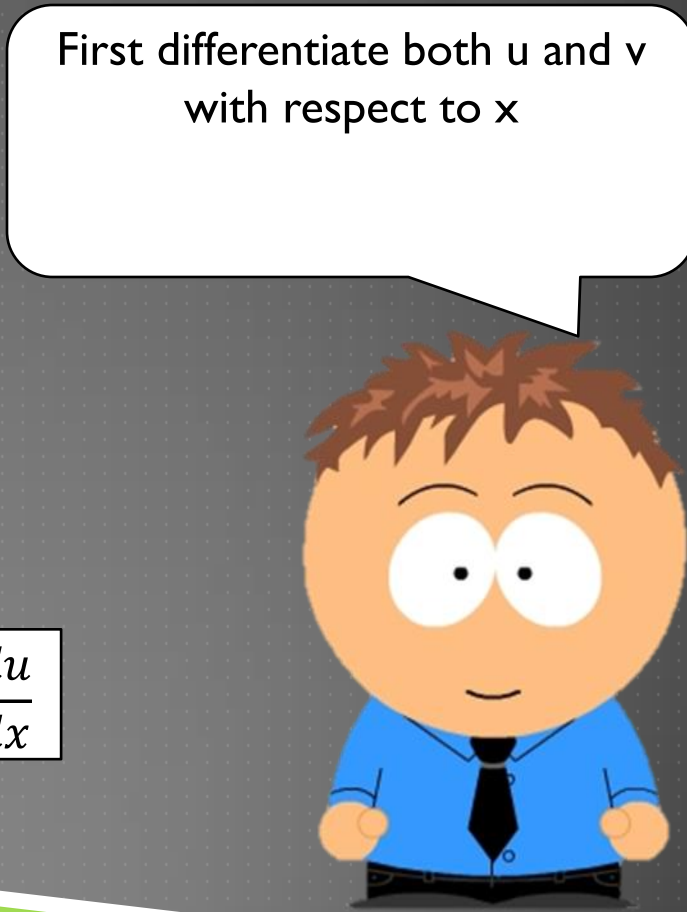
$u = e^{3x}$ $\qquad v = \sin(x)$



$$\boxed{y = uv \;\Rightarrow\; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

THE PRODUCT RULE

Differentiating when one function is multiplied by another

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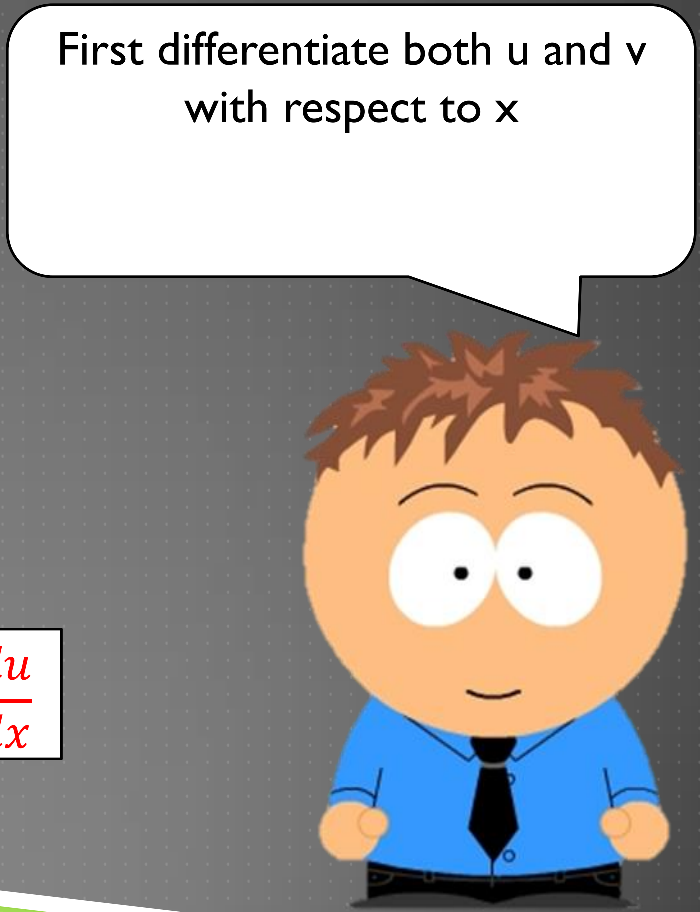
$u = e^{3x}$ $\quad\quad$ $v = \sin(x)$



$$\boxed{y = uv \quad \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

THE PRODUCT RULE

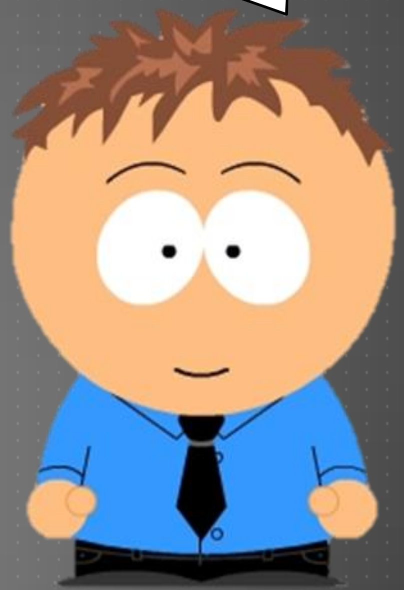
Differentiating when one function is multiplied by another

$f(x) = e^{3x}\sin(x)$

$u = e^{3x}$ $\qquad$ $v = \sin(x)$

$\frac{du}{dx} = 3e^{3x}$ $\qquad$ $\frac{dv}{dx} = \cos(x)$

First differentiate both u and v with respect to x



$$\boxed{y = uv \;\Rightarrow\; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

THE PRODUCT RULE

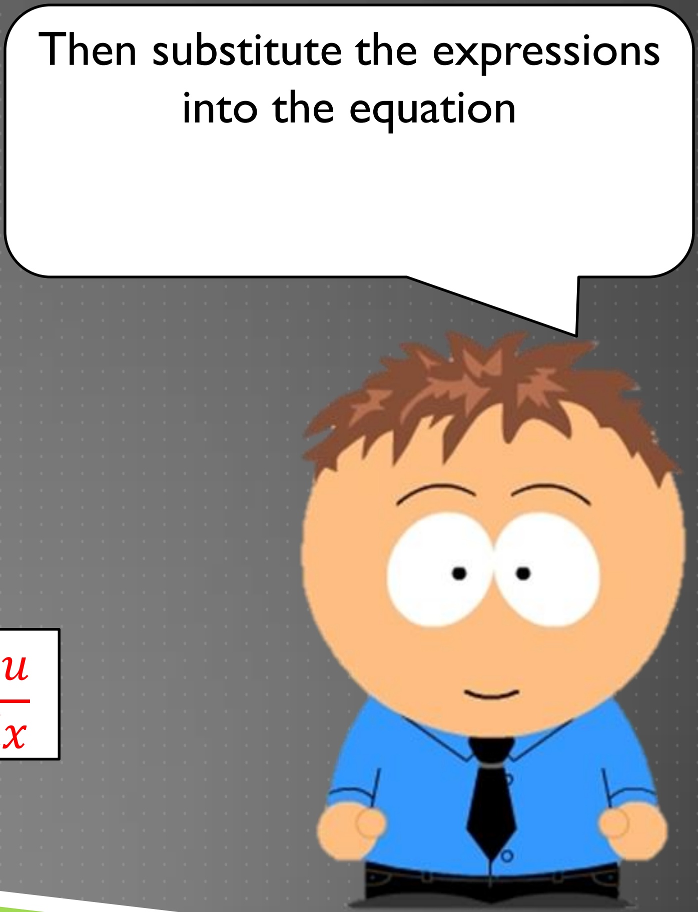
<u>Differentiating when one function is multiplied by another</u>

$$f(x) = e^{3x}\sin(x)$$

$$u = e^{3x} \qquad v = \sin(x)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{du}{dx} = 3e^{3x} \qquad \frac{dv}{dx} = \cos(x)$$



$$\boxed{y = uv \;\; \Rightarrow \;\; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

THE PRODUCT RULE

Differentiating when one function is multiplied by another

$$f(x) = e^{3x}\sin(x)$$

$$u = e^{3x} \qquad v = \sin(x)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{du}{dx} = 3e^{3x} \qquad \frac{dv}{dx} = \cos(x)$$

$$f'(x) = e^{3x} \times \cos(x) + \sin(x) \times 3e^{3x}$$

$$\boxed{y = uv \;\Rightarrow\; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Then substitute the expressions into the equation

THE PRODUCT RULE

Differentiating when one function is multiplied by another

$$f(x) = e^{3x}\sin(x)$$

$$u = e^{3x} \qquad v = \sin(x)$$

$$\downarrow \qquad\qquad \downarrow$$

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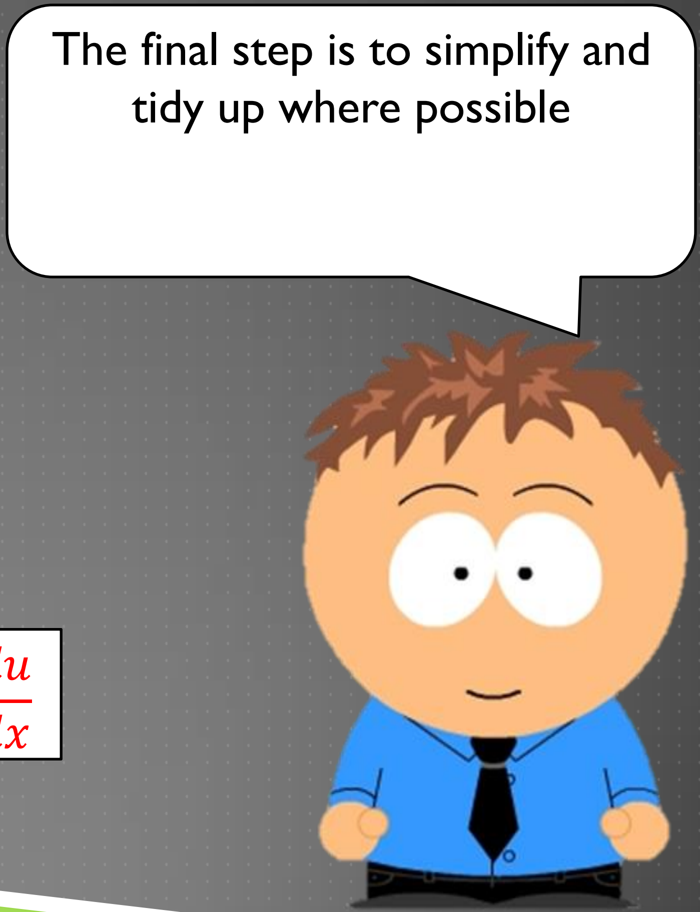

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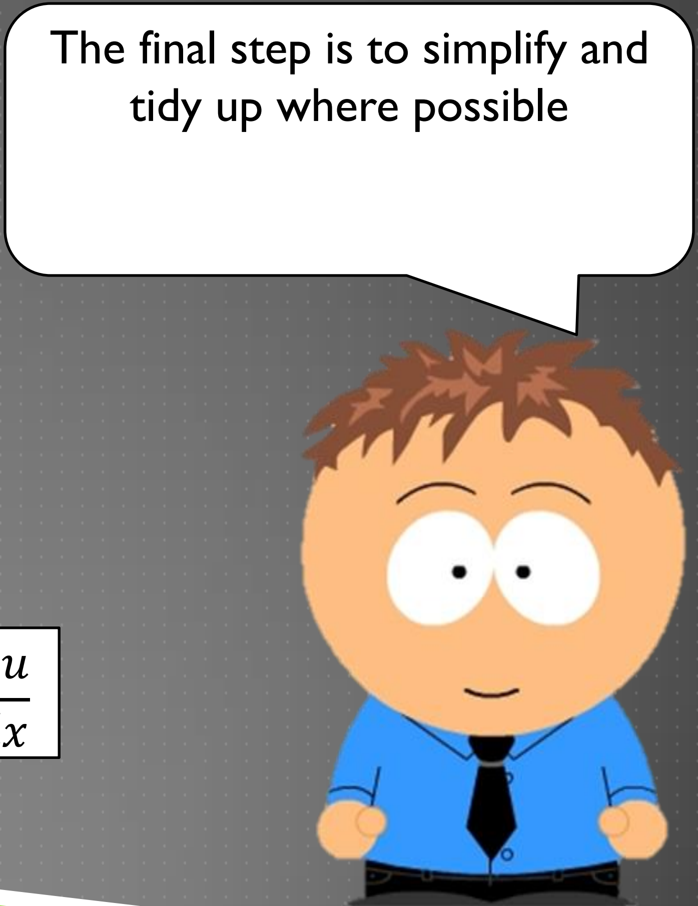
$$f'(x) = e^{3x} \times \cos(x) + \sin(x) \times 3e^{3x}$$

$$f'(x) = e^{3x}\,(\cos(x) + 3\sin(x))$$

$$\boxed{y = uv \;\Rightarrow\; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$



THE QUOTIENT RULE

Differentiating when one function is divided by another

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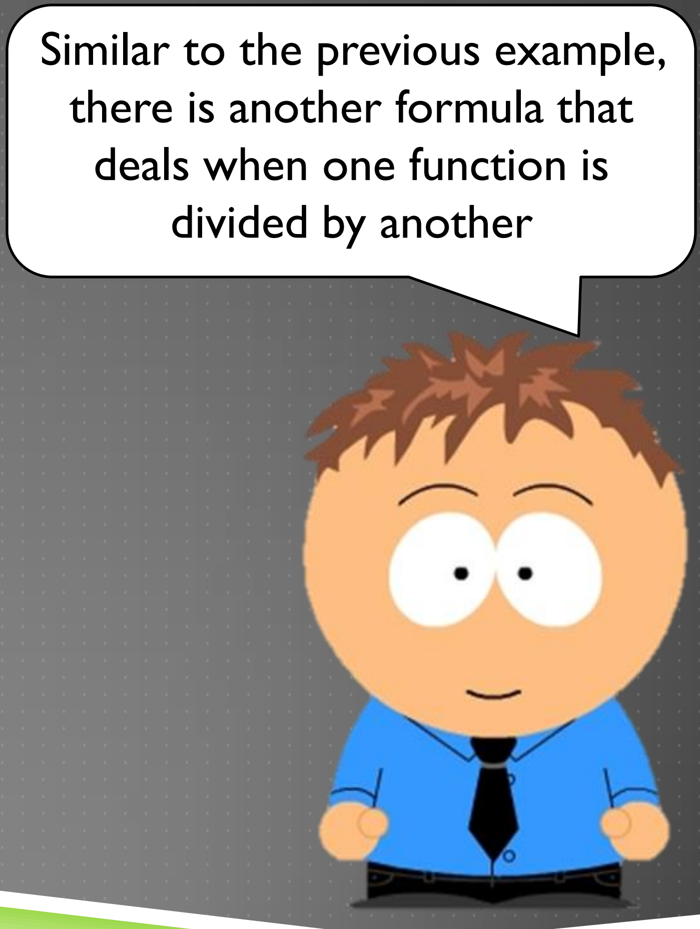
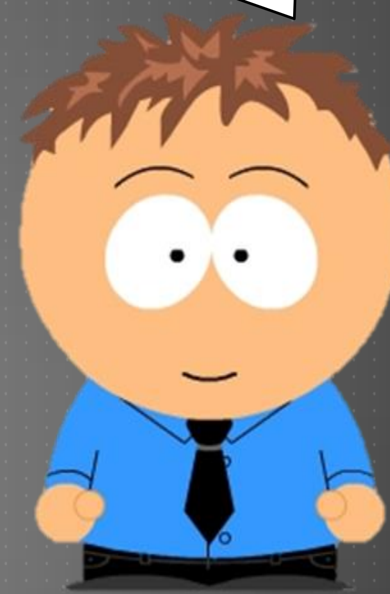


THE QUOTIENT RULE

Differentiating when one function is divided by another

For example, take the function

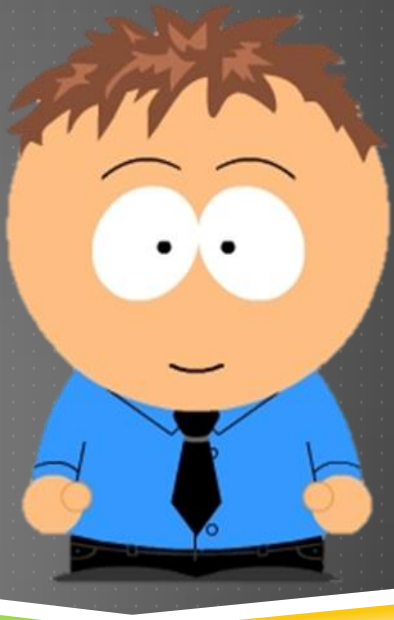
$$\frac{x^2 - 4x + 12}{(x-3)^2}$$



THE QUOTIENT RULE

Differentiating when one function is divided by another

$$f(x) = \frac{x^2-4x+12}{(x-3)^2}$$



THE QUOTIENT RULE

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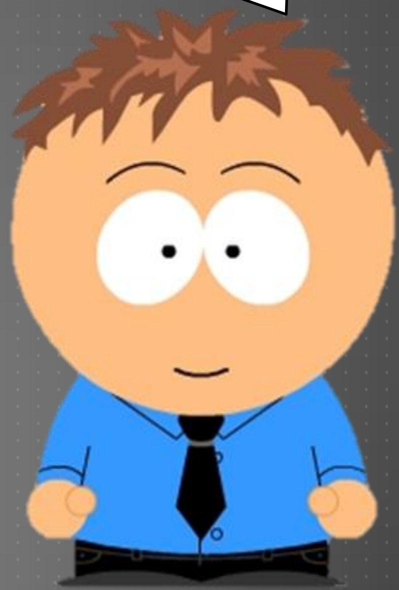
$$f(x) = \frac{x^2 - 4x + 12}{(x-3)^2}$$

Again, label u and v, and find the derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$



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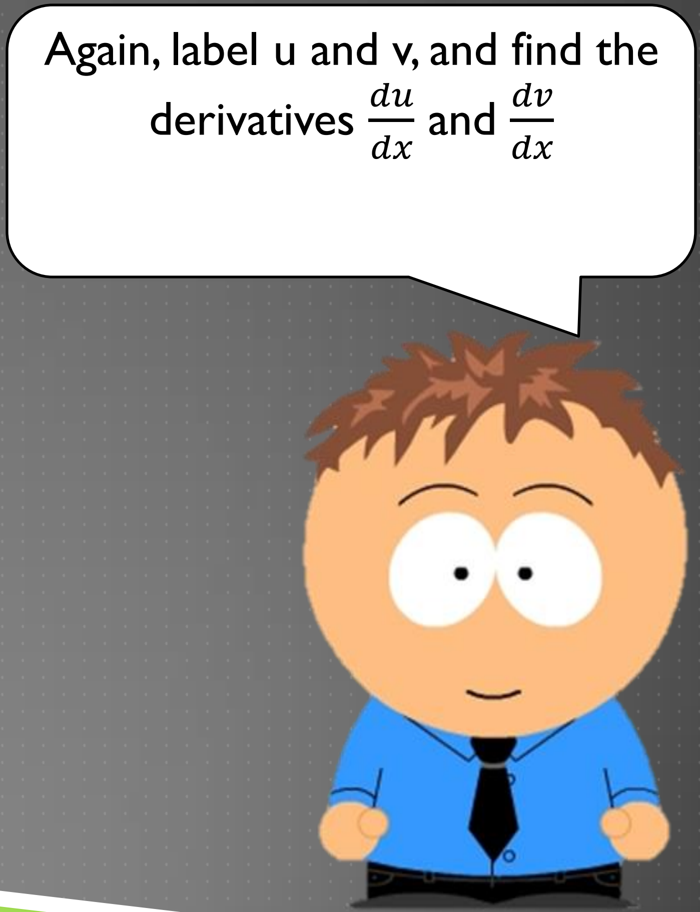
$$f(x) = \frac{x^2 - 4x + 12}{(x-3)^2}$$

$$u = x^2 - 4x + 12$$



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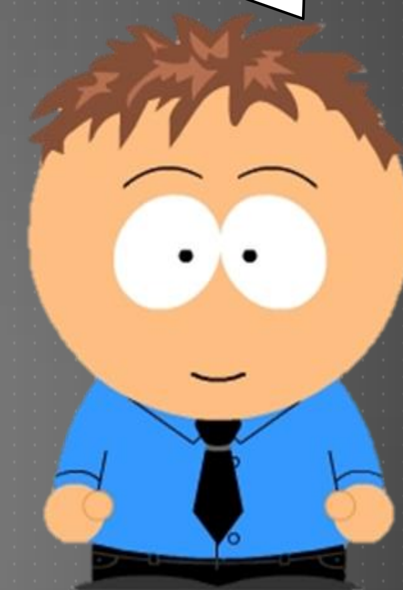
$$f(x) = \frac{x^2-4x+12}{(x-3)^2}$$

$$u = x^2 - 4x + 12 \qquad v = (x-3)^2$$

Again, label u and v, and find the derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$



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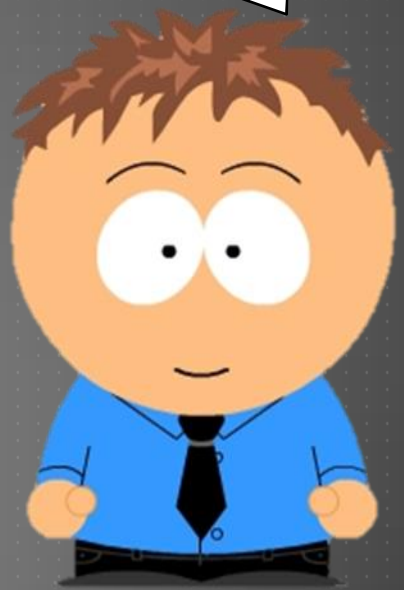
$$f(x) = \frac{x^2-4x+12}{(x-3)^2}$$

$u = x^2 - 4x + 12$ $\qquad$ $v = (x-3)^2$

$$\frac{du}{dx} = 2x - 4$$

Again, label u and v, and find the derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$



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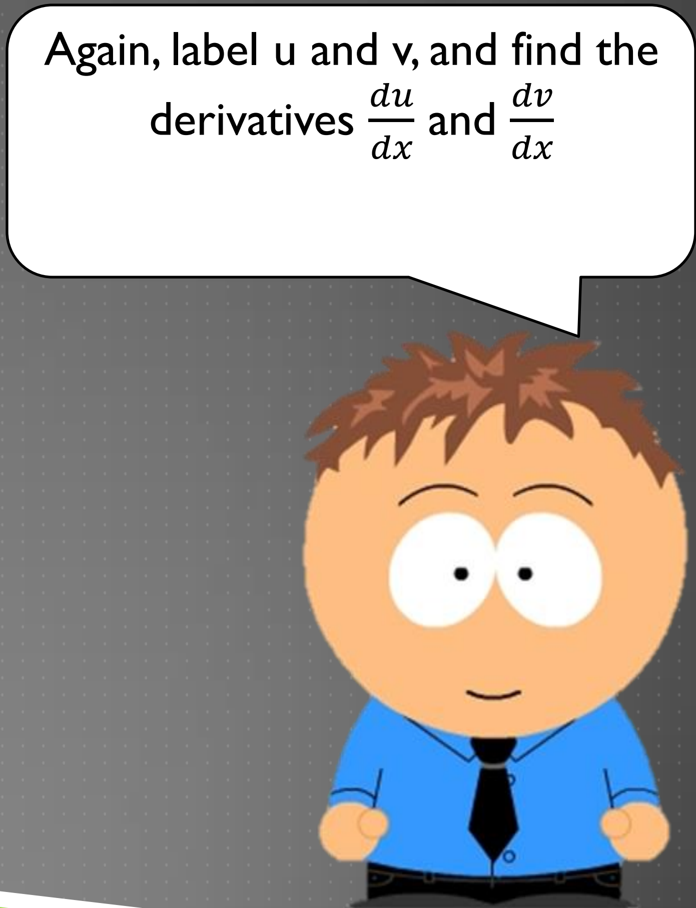

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This time we substitute into the formula:

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

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$$f'(x) = \frac{(x-3)^2 \times (2x-4) - (x^2-4x+12) \times 2(x-3)}{[(x-3)^2]^2}$$

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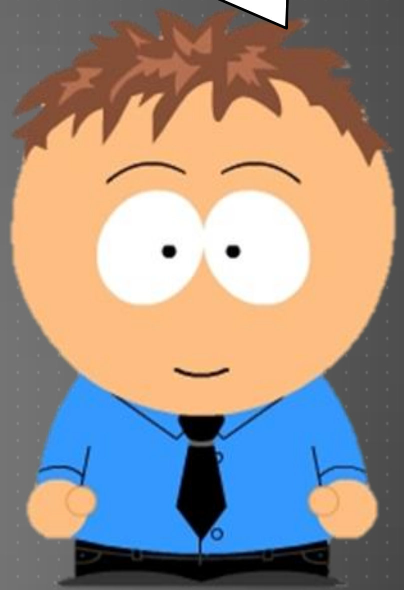

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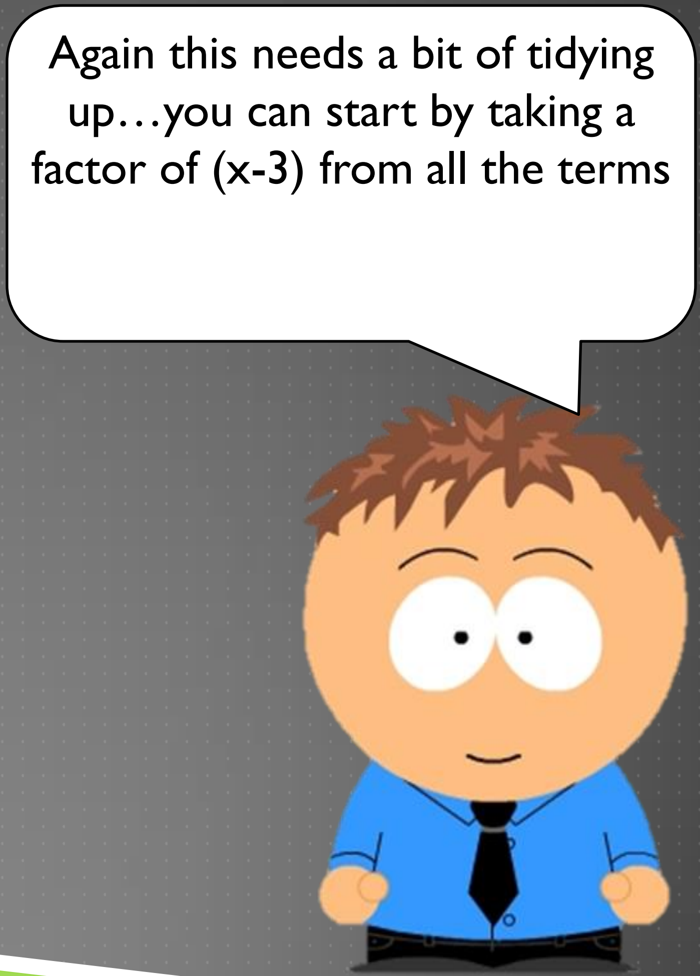

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$$f'(x) = \frac{(x-3)^2 \times (2x-4) - (x^2 - 4x + 12) \times 2(x-3)}{[(x-3)^2]^2}$$

$$= \frac{(x-3) \times (2x-4) - (x^2 - 4x + 12) \times 2}{(x-3)^3}$$

Again this needs a bit of tidying up…you can start by taking a factor of (x-3) from all the terms

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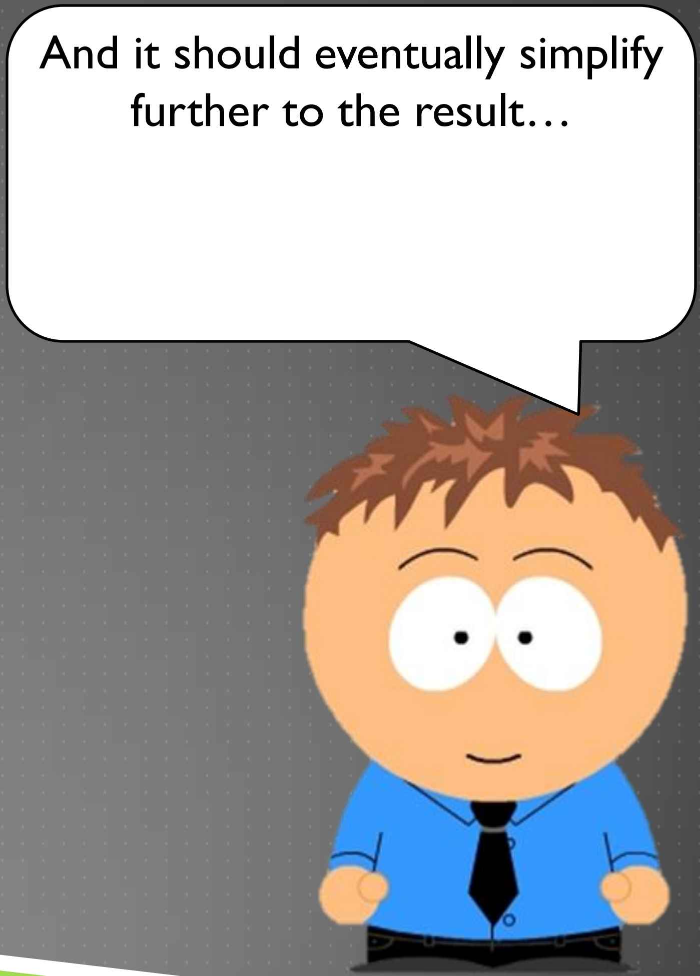

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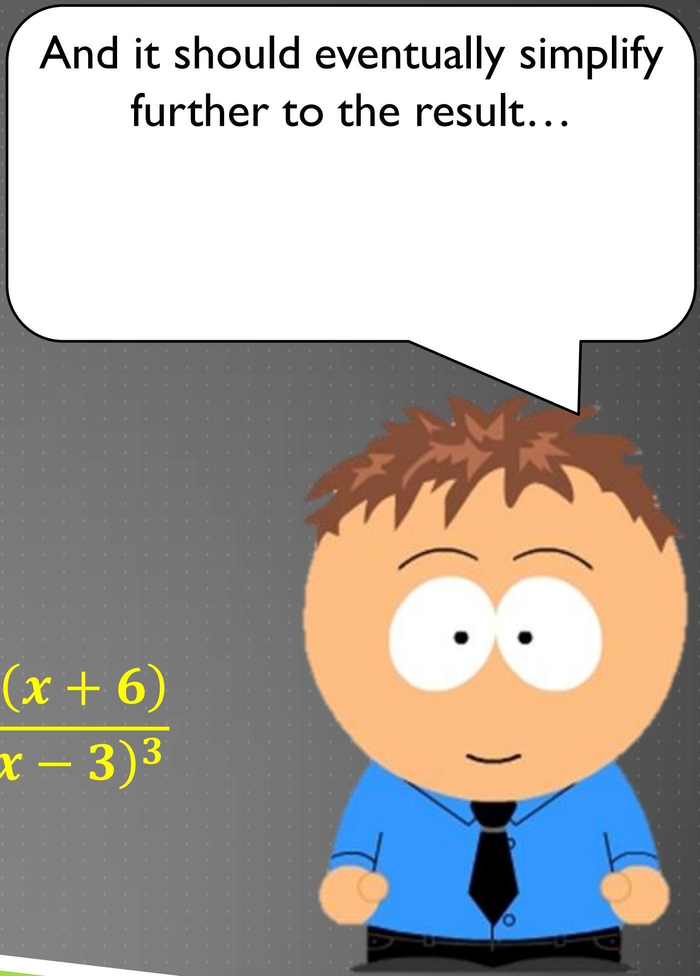
$$f'(x) = \frac{(x-3)^2 \times (2x-4) - (x^2-4x+12) \times 2(x-3)}{[(x-3)^2]^2}$$

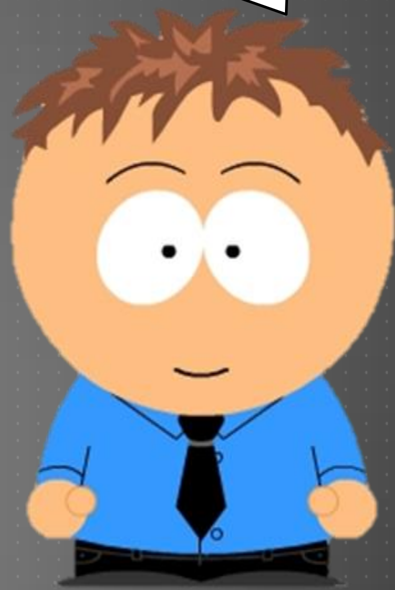
$$= \frac{(x-3) \times (2x-4) - (x^2-4x+12) \times 2}{(x-3)^3} \quad = -\frac{\mathbf{2(x+6)}}{\mathbf{(x-3)^3}}$$



INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

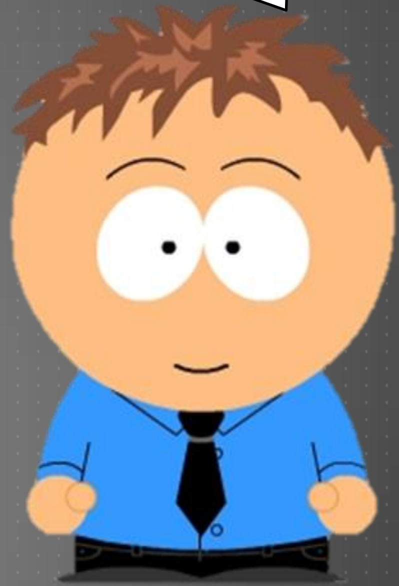
Previously we saw how we could try to reverse the chain rule, and integrate functions like $(2x-3)^4$ *by inspection*



INTEGRATION BY SUBSTITUTION

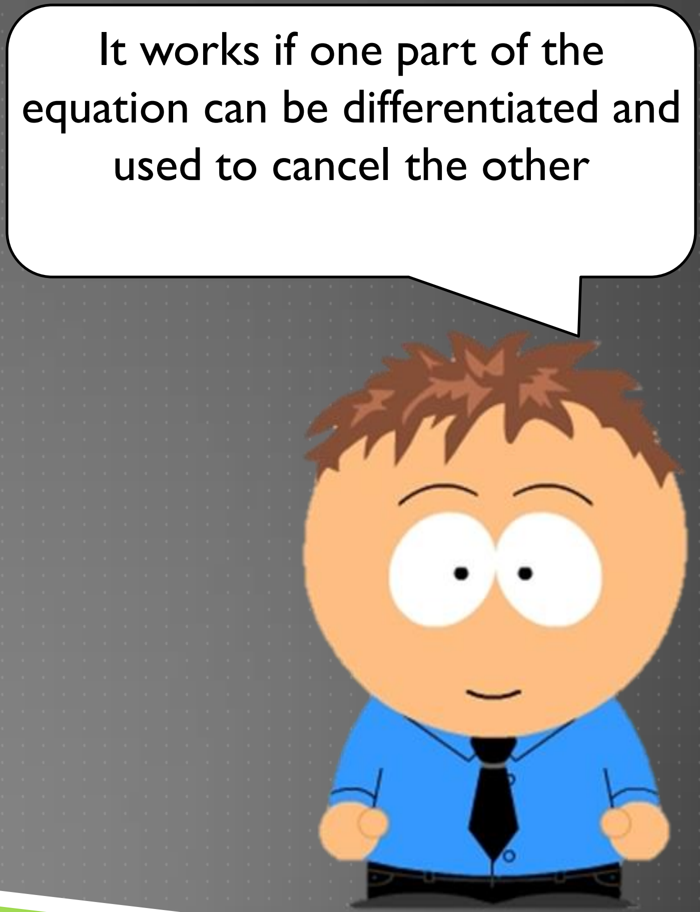
<u>Integrating more complex expressions</u>

This idea can be extended to some other products and quotients, using a method called *substitution*



INTEGRATION BY SUBSTITUTION

Integrating more complex expressions



INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

Take the example

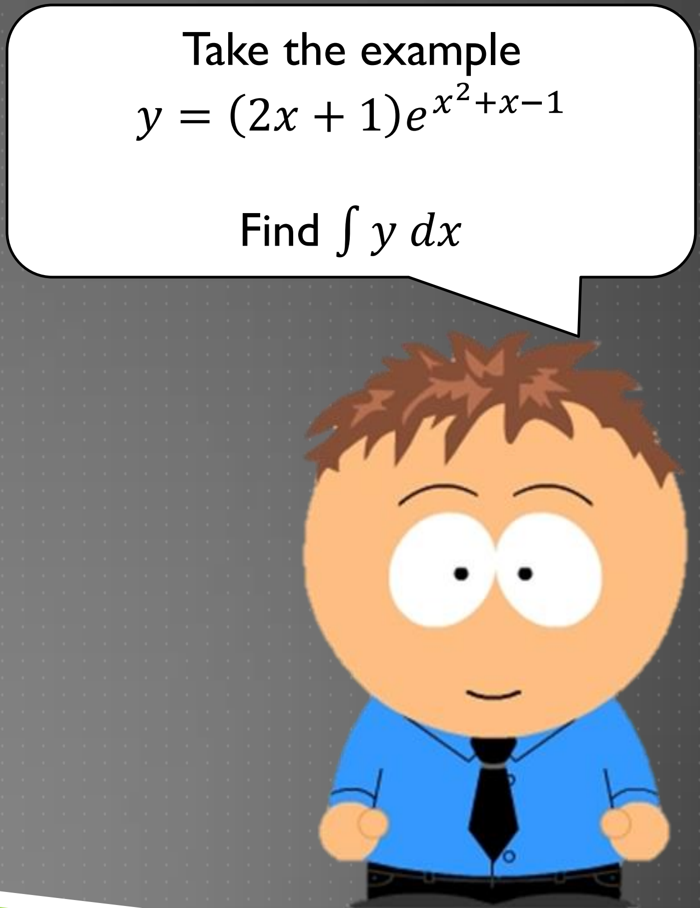
$$y = (2x + 1)e^{x^2+x-1}$$

Find $\int y \, dx$

INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

$\int (2x+1)e^{x^2+x-1} \; dx$

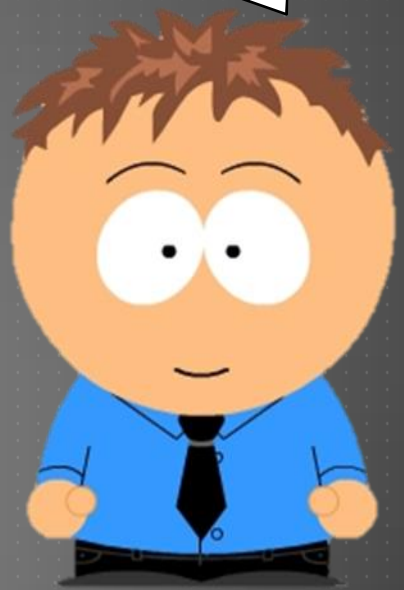


INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

$$\int (2x+1)e^{x^2+x-1} \; dx$$

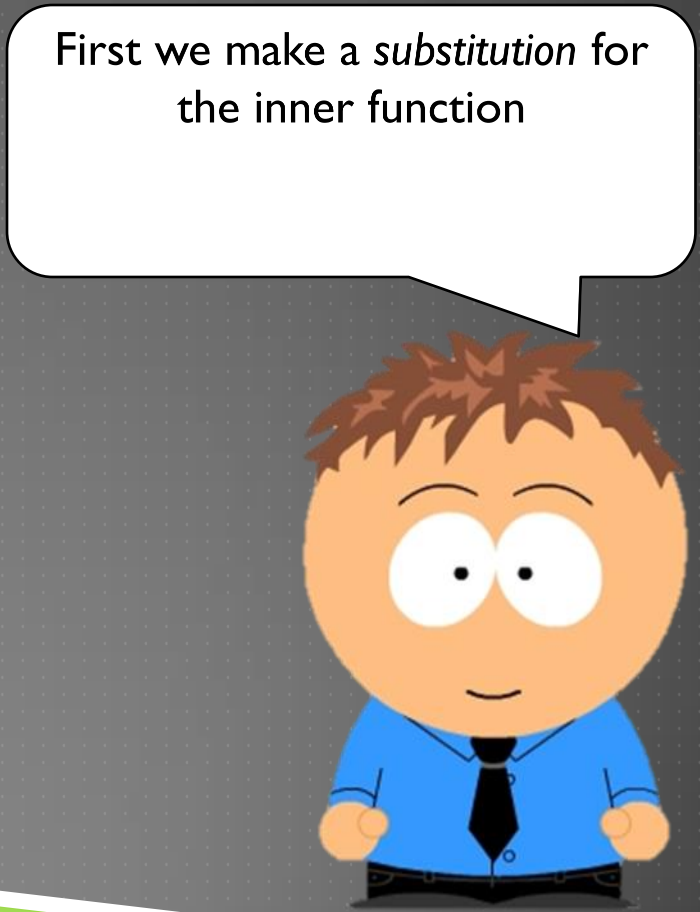
The *inner* part of the composite function e^{x^2+x-1}
Differentiates to give the second function, $2x+1$



INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

$$\int (2x+1)e^{x^2+x-1}\ dx$$



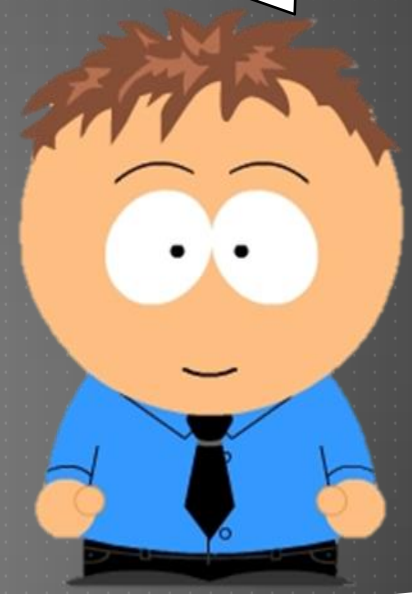
INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

$\int (2x+1)e^{x^2+x-1} \; dx$

Let $u = x^2 + x - 1$

First we make a *substitution,* for the inner function

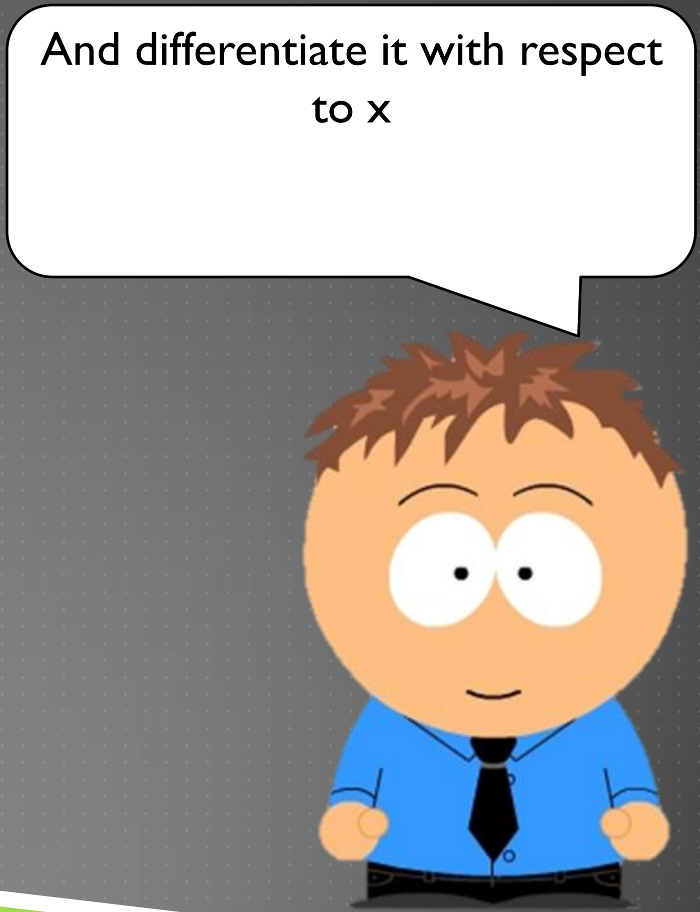


INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

$\int (2x+1)e^{x^2+x-1} \ dx$

Let $u = x^2 + x - 1$



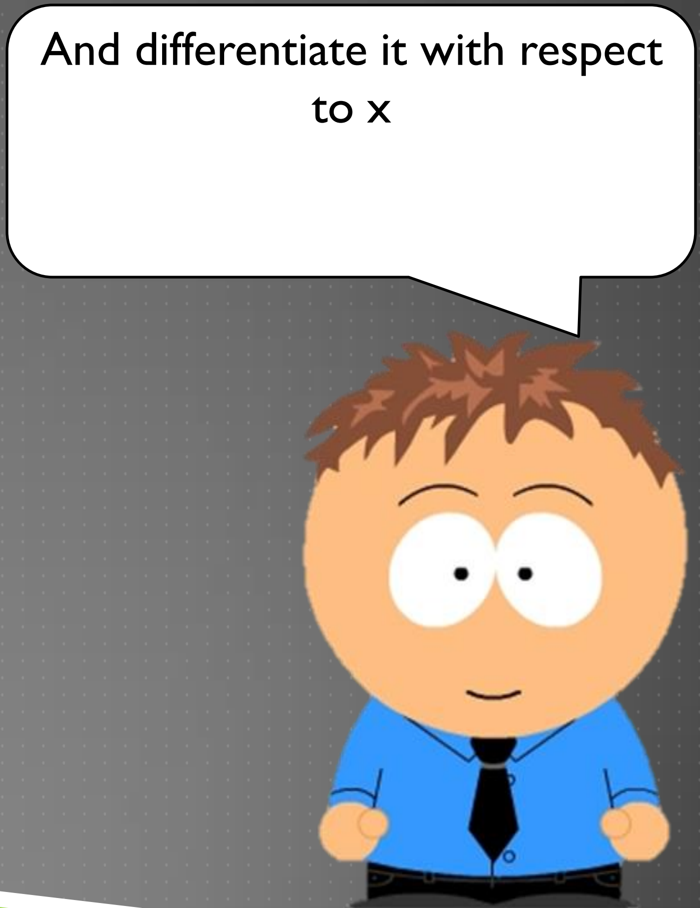
INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

$$\int (2x+1)e^{x^2+x-1} \; dx$$

Let $u = x^2 + x - 1$

$$\frac{du}{dx} = 2x + 1$$



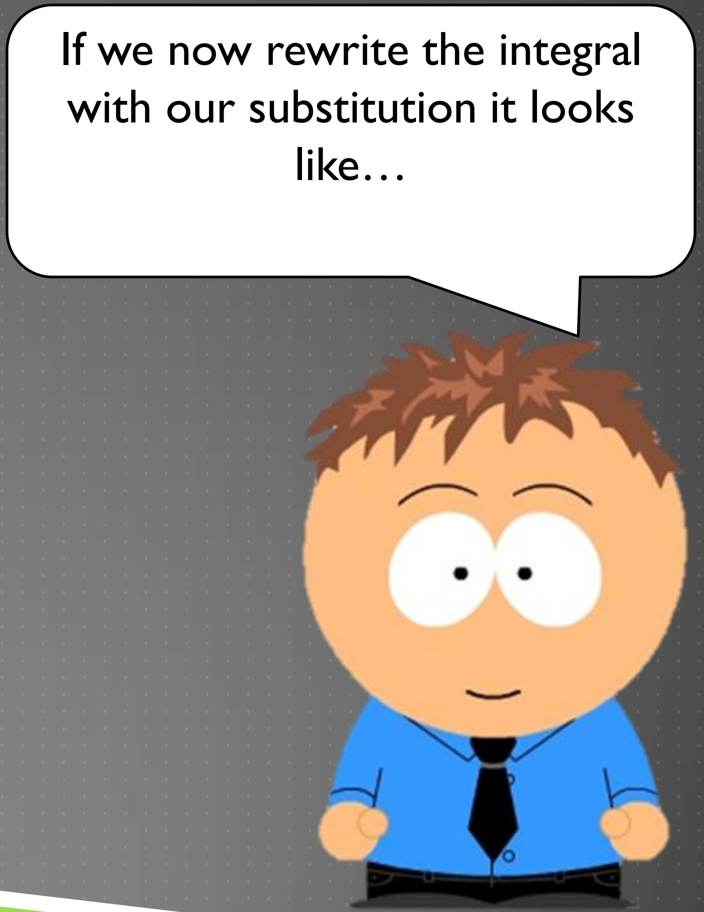
INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

$\int (2x+1)e^{x^2+x-1} \; dx$

Let $u = x^2 + x - 1$

$\frac{du}{dx} = 2x + 1$



INTEGRATION BY SUBSTITUTION

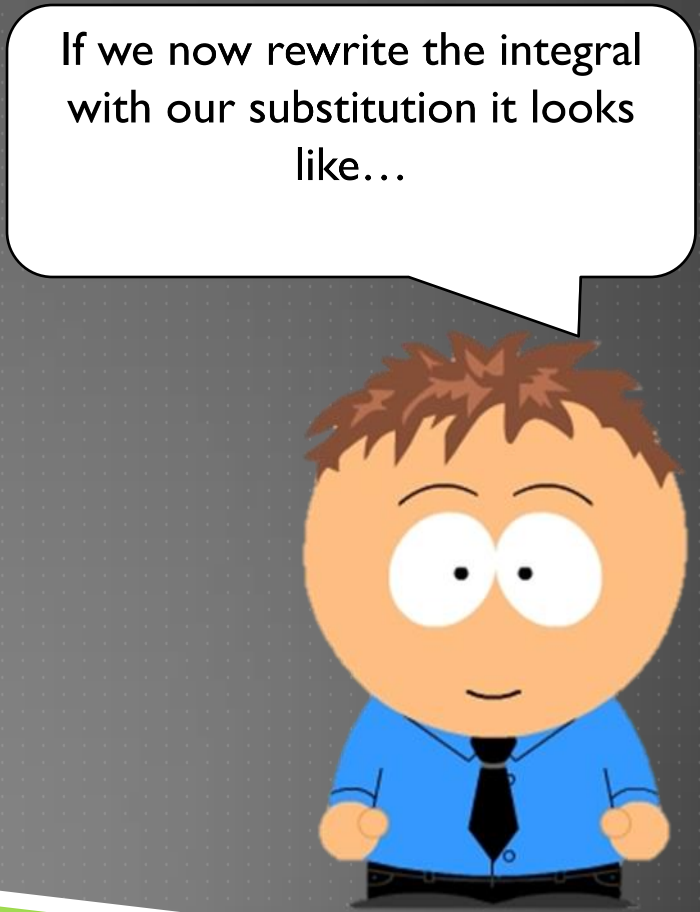
<u>Integrating more complex expressions</u>

$\int (2x+1)e^{x^2+x-1} \; dx$

Let $u = x^2 + x - 1$

$\frac{du}{dx} = 2x + 1$

$= \int (2x+1)e^{u} \; dx$



INTEGRATION BY SUBSTITUTION

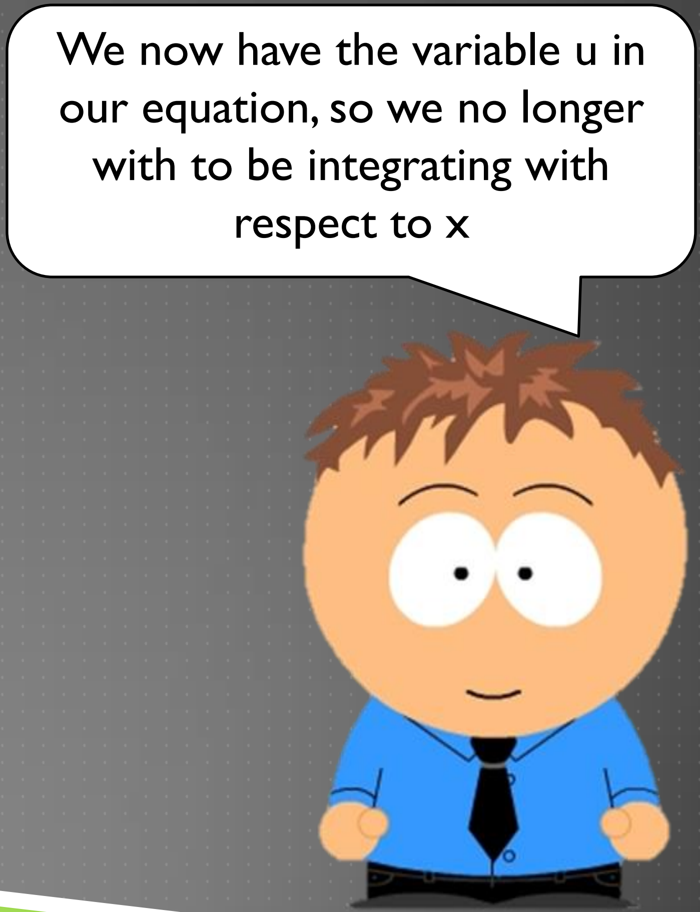
Integrating more complex expressions

$\int (2x+1)e^{x^2+x-1} \ dx$

Let $u = x^2 + x - 1$

$\frac{du}{dx} = 2x + 1$

$= \int (2x+1)e^{u} \ dx$



INTEGRATION BY SUBSTITUTION

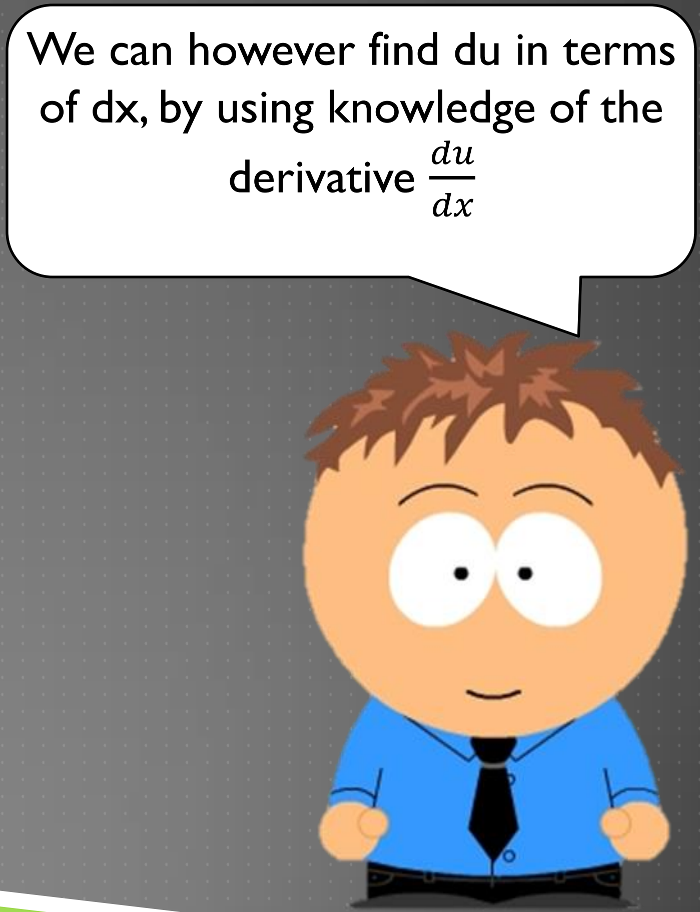
Integrating more complex expressions

$\int (2x+1)e^{x^2+x-1} \; dx$

Let $u = x^2 + x - 1$

$\frac{du}{dx} = 2x + 1$

$= \int (2x+1)e^{u} \; dx$



INTEGRATION BY SUBSTITUTION

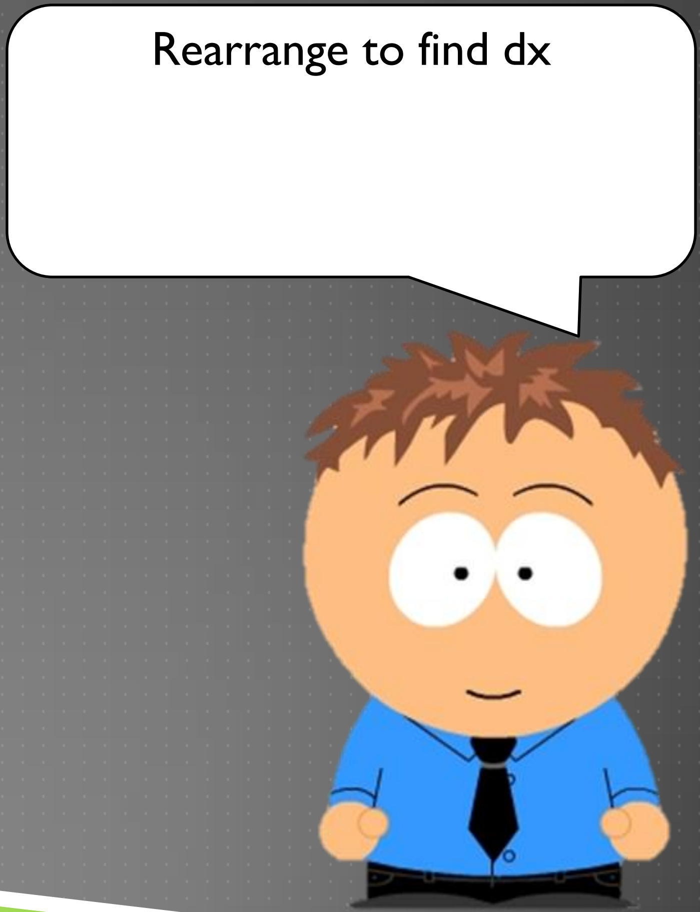
<u>Integrating more complex expressions</u>

$$\int (2x+1)e^{x^2+x-1}\ \ dx$$

Let $u = x^2 + x - 1$

$$\frac{du}{dx} = 2x + 1$$

$$= \int (2x+1)e^{u}\ \ dx$$



INTEGRATION BY SUBSTITUTION

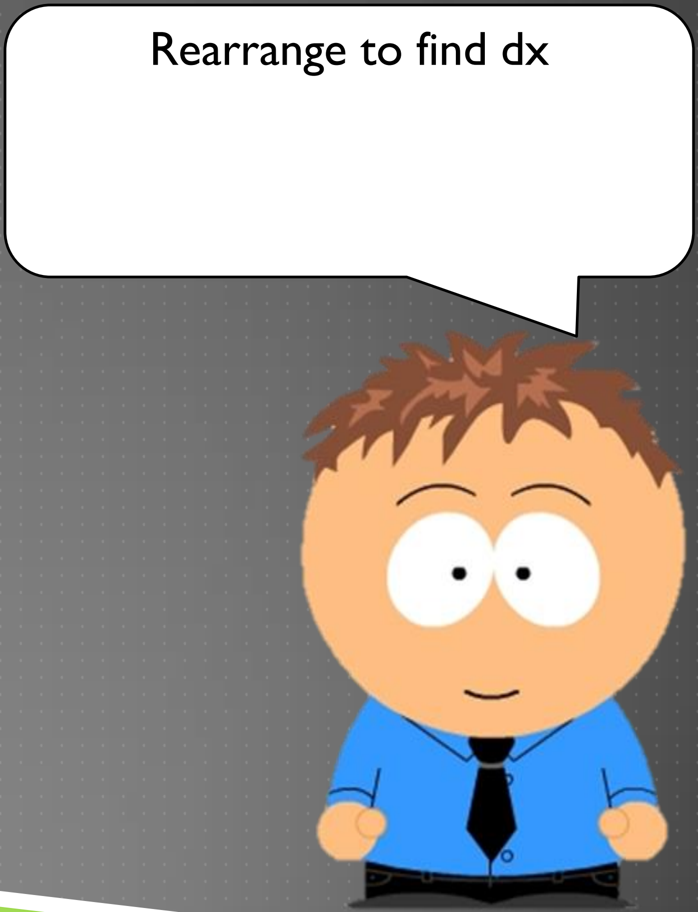
Integrating more complex expressions

$\int (2x+1)e^{x^2+x-1} \; dx$

Let $u = x^2 + x - 1$

$\frac{du}{dx} = 2x + 1 \quad \Longrightarrow \quad \frac{du}{2x+1} = dx$

$= \int (2x+1)e^{u} \; dx$



INTEGRATION BY SUBSTITUTION

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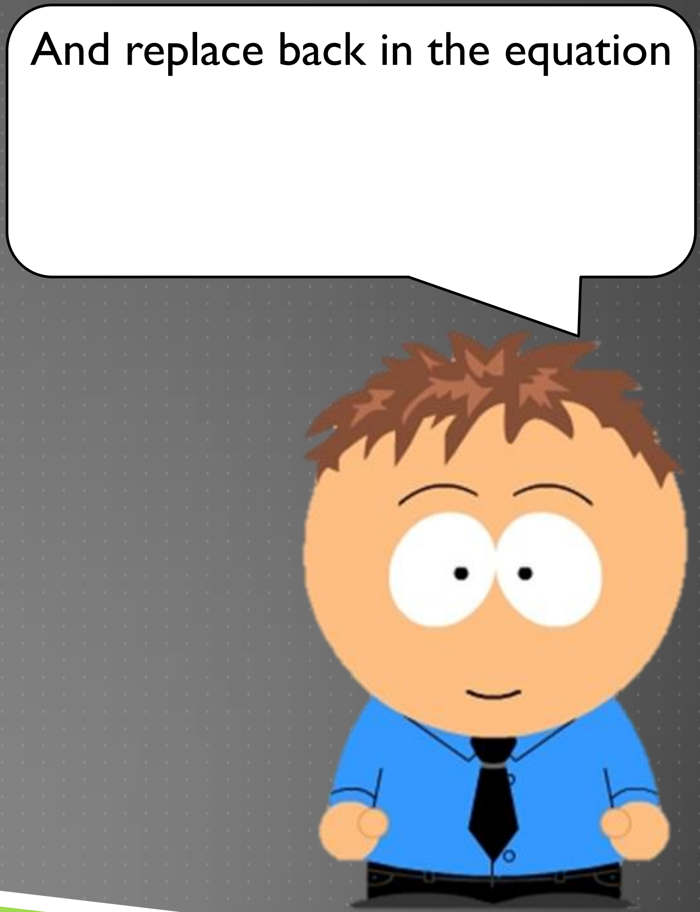

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

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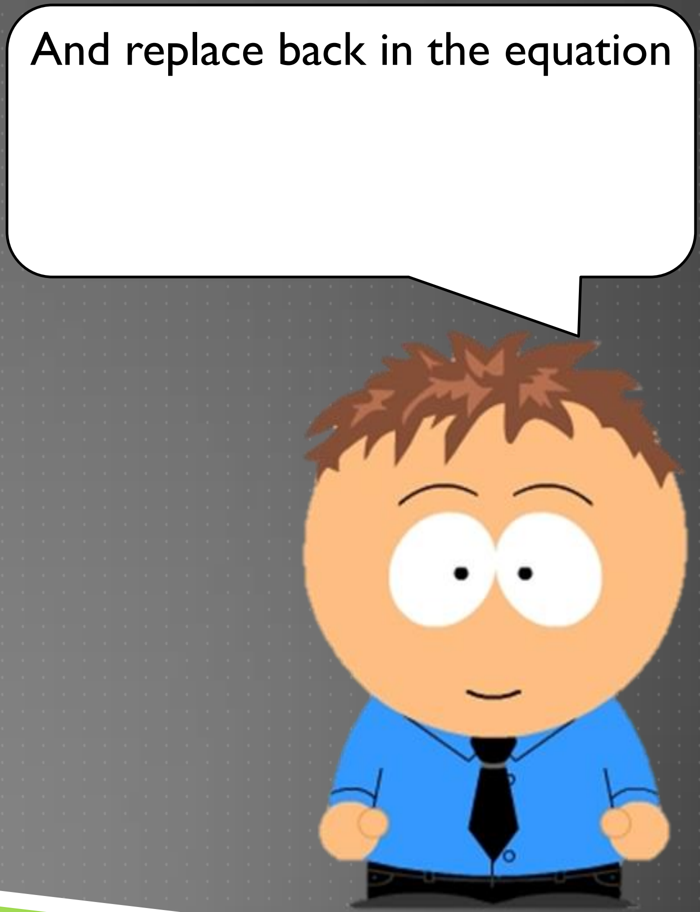
$= \int (2x+1)e^{u} \; dx$

$= \int (2x+1)e^{u} \, \frac{du}{2x+1}$



INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

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Let $u = x^2 + x - 1$

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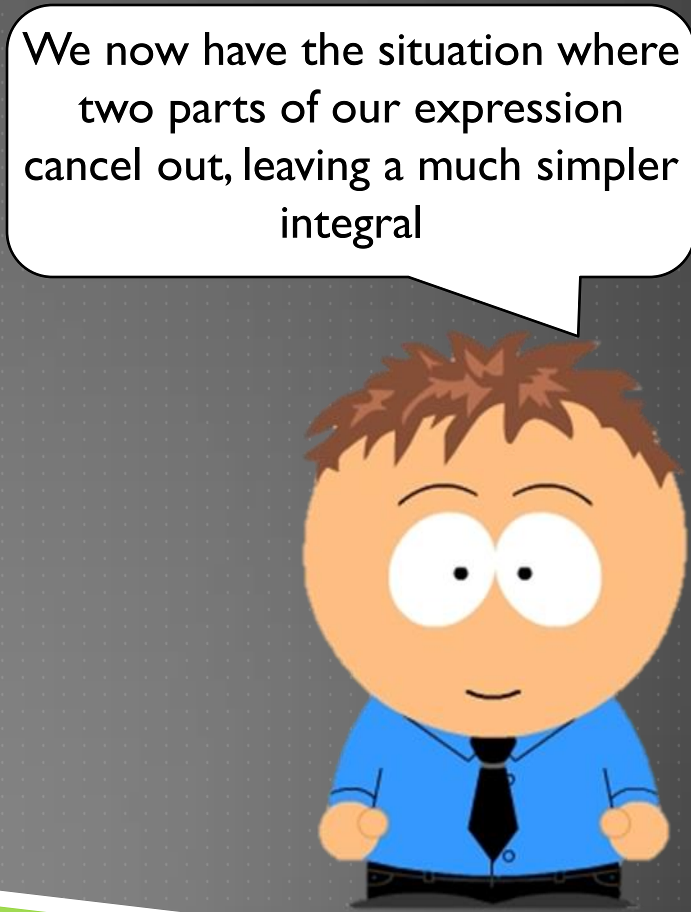

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

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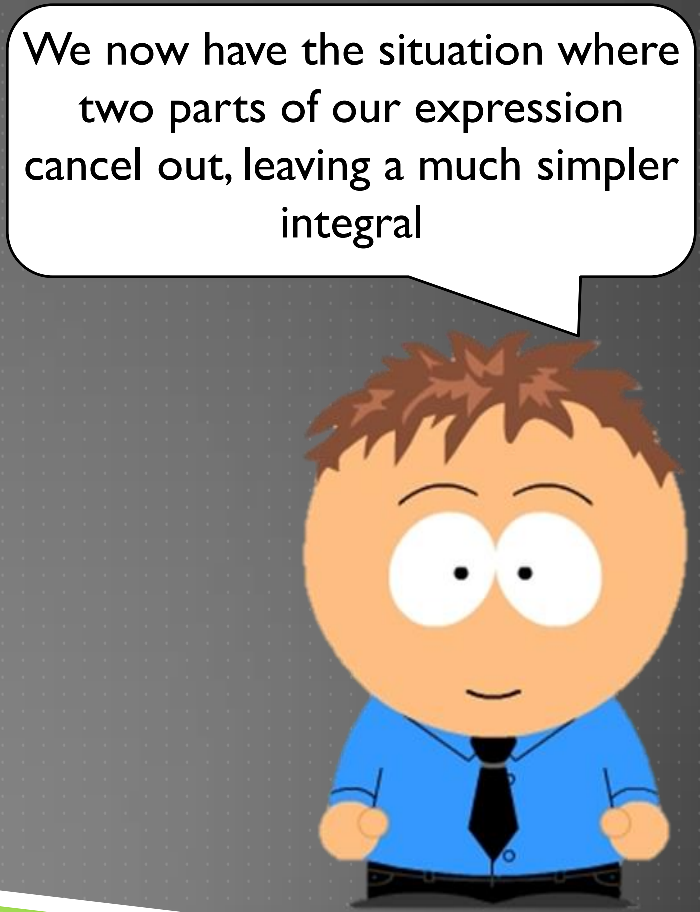

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

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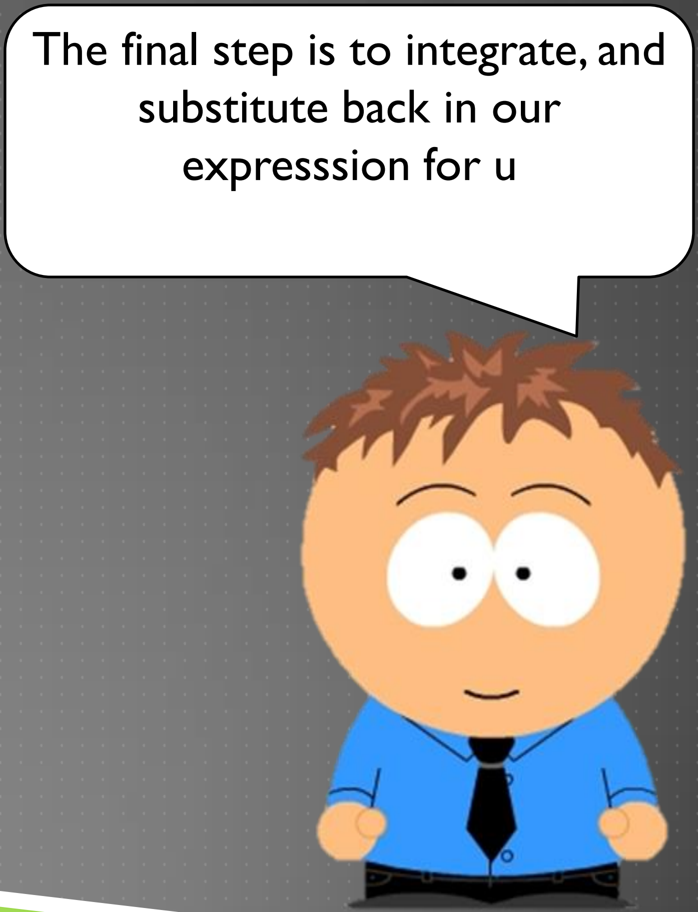

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

$$\int (2x+1)e^{x^2+x-1}\ dx$$

Let $u = x^2 + x - 1$

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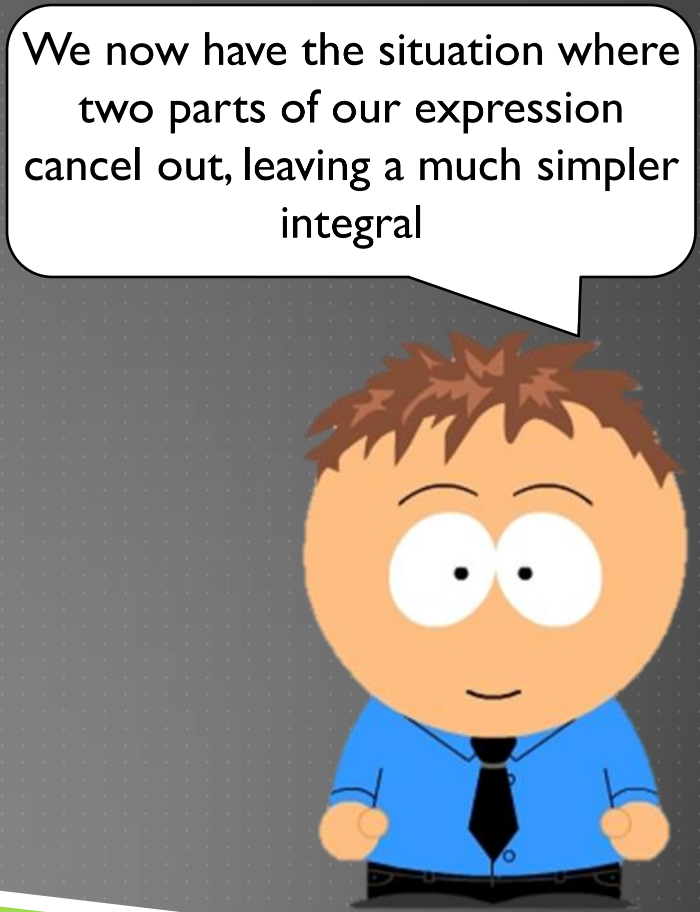
$$= \int (2x+1)e^{u}\ dx$$

$$= \int (2x+1)e^{u}\ \frac{du}{2x+1}$$

$$= \int e^{u}\ du$$

$$= e^{u} + c$$



INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

$$\int (2x+1)e^{x^2+x-1} \; dx$$

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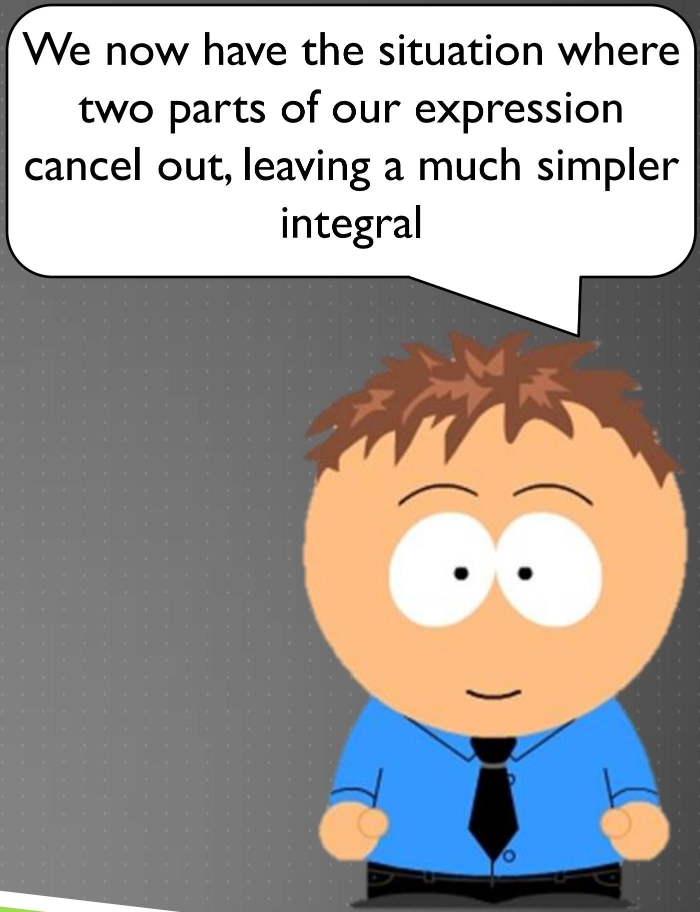
$$= \int (2x+1)e^{u} \frac{du}{2x+1}$$

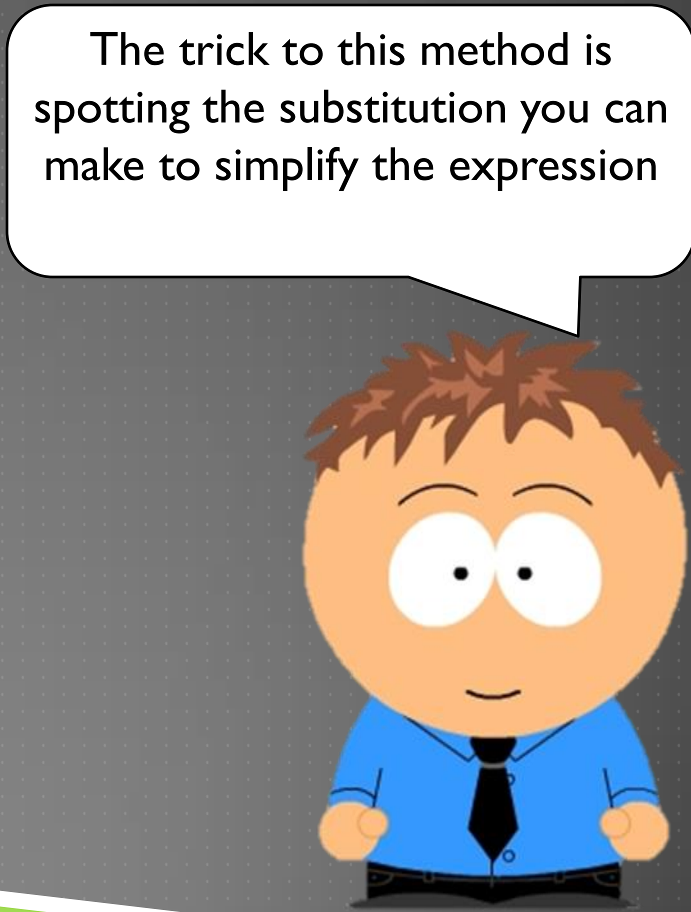
$$= \int e^{u} \; du$$

$$= e^{u} + c \qquad \boldsymbol{= e^{x^2+x-1} + c}$$



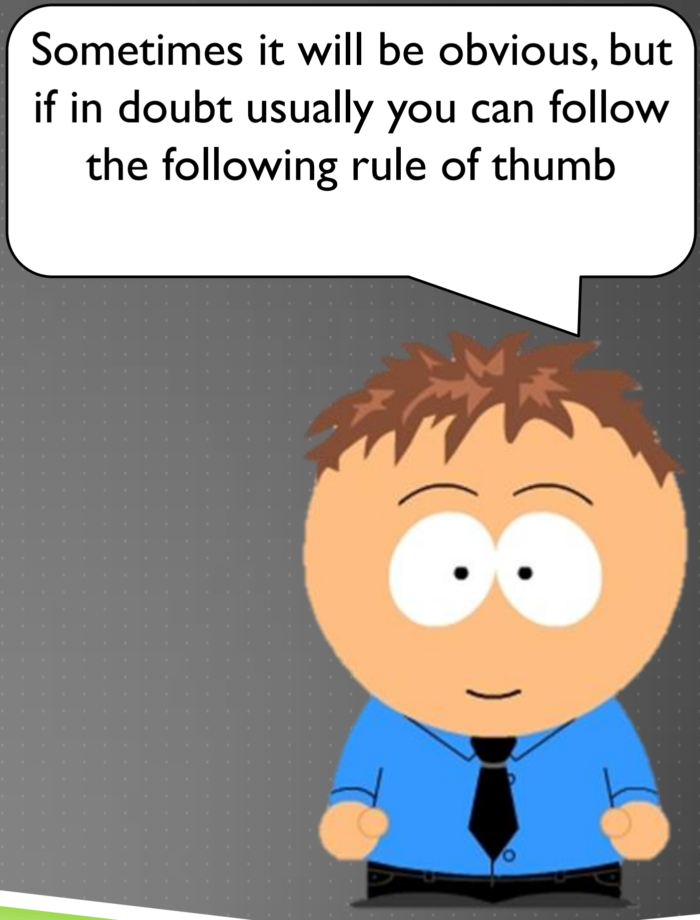
INTEGRATION BY SUBSTITUTION

Integrating more complex expressions



INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

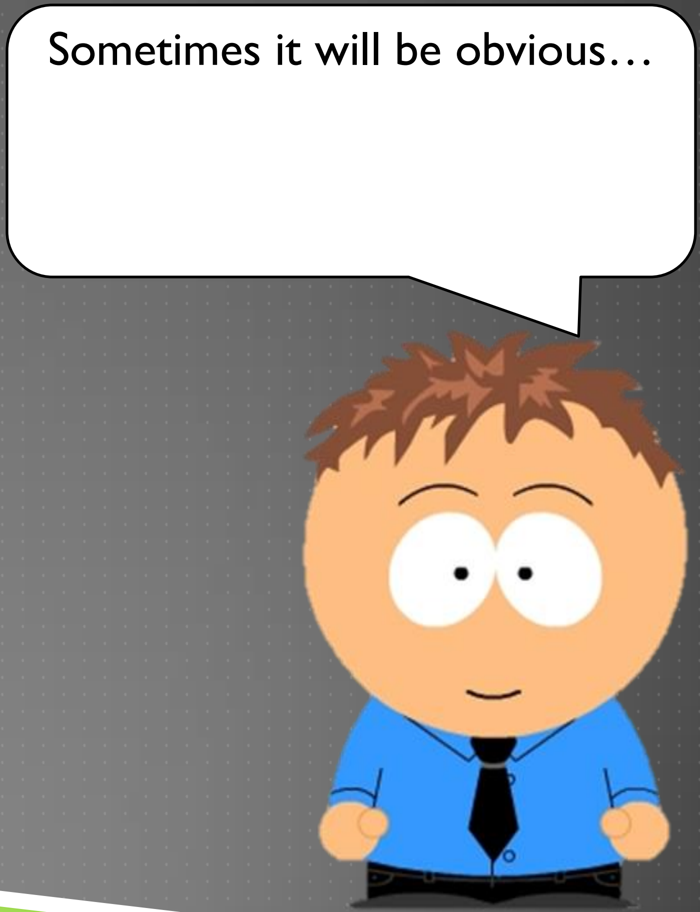


INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

Examples

Expression	**Substitution**
$\int x(x^2+3)^3 \ dx$	
$\int \frac{x^2}{5x^3-1} \ dx$	
$\int 3xe^{x^2} \ dx$	
$\int \frac{e^x}{\sqrt{e^x+1}} \ dx$	



INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

Examples

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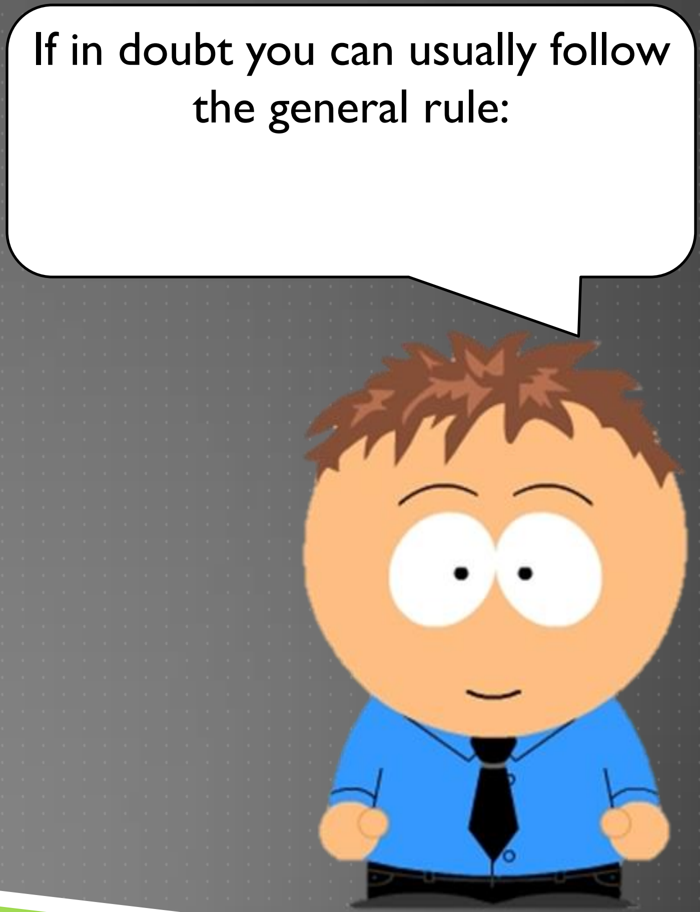

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

Examples

Expression	Substitution
$\int x(x^2+3)^3\ dx$	
$\int \frac{x^2}{5x^3-1}\ dx$	
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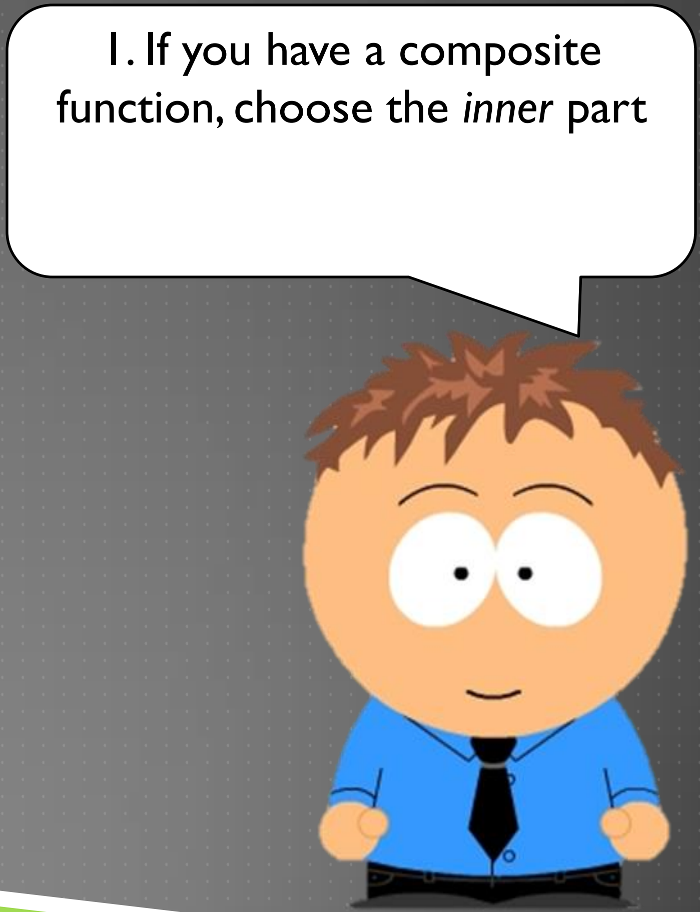


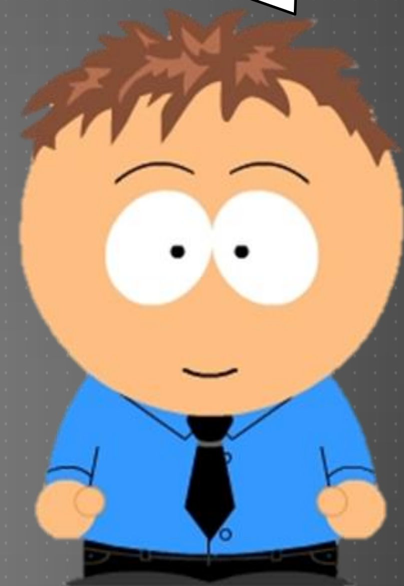
INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

Examples

Expression	**Substitution**
$\int x(x^2+3)^3\ dx$	
$\int \frac{x^2}{5x^3-1}\ dx$	
$\int 3xe^{x^2}\ dx$	
$\int \frac{e^x}{\sqrt{e^x+1}}\ dx$	

1. If you have a composite function, choose the *inner* part



INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

Examples

Expression	**Substitution**
$\int x(x^2+3)^3\ dx$	$u = x^2 + 3$
$\int \frac{x^2}{5x^3-1}\ dx$	
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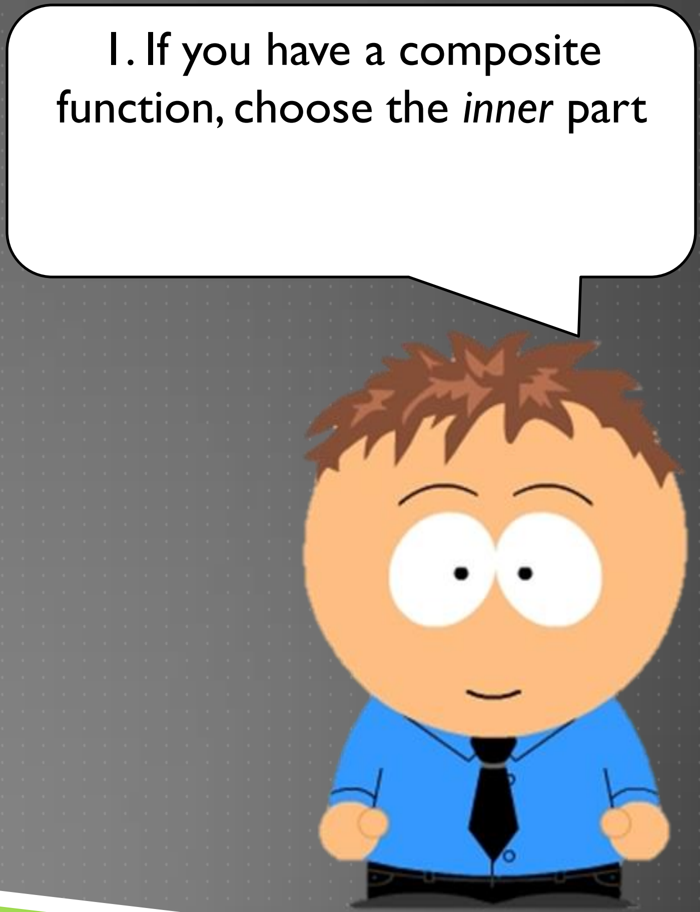

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

Examples

Expression	**Substitution**
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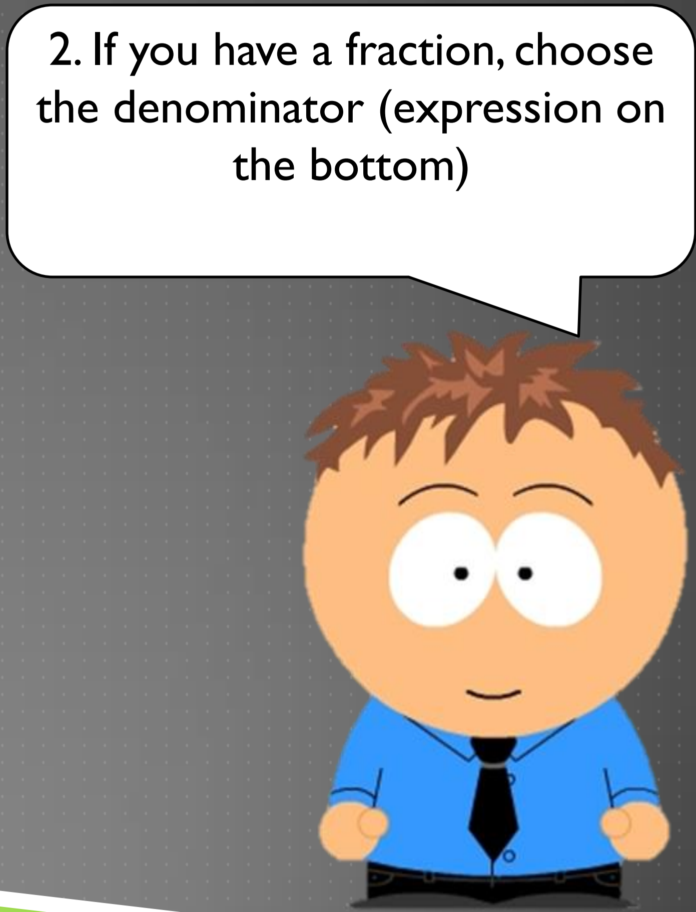

INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

Examples

Expression	Substitution
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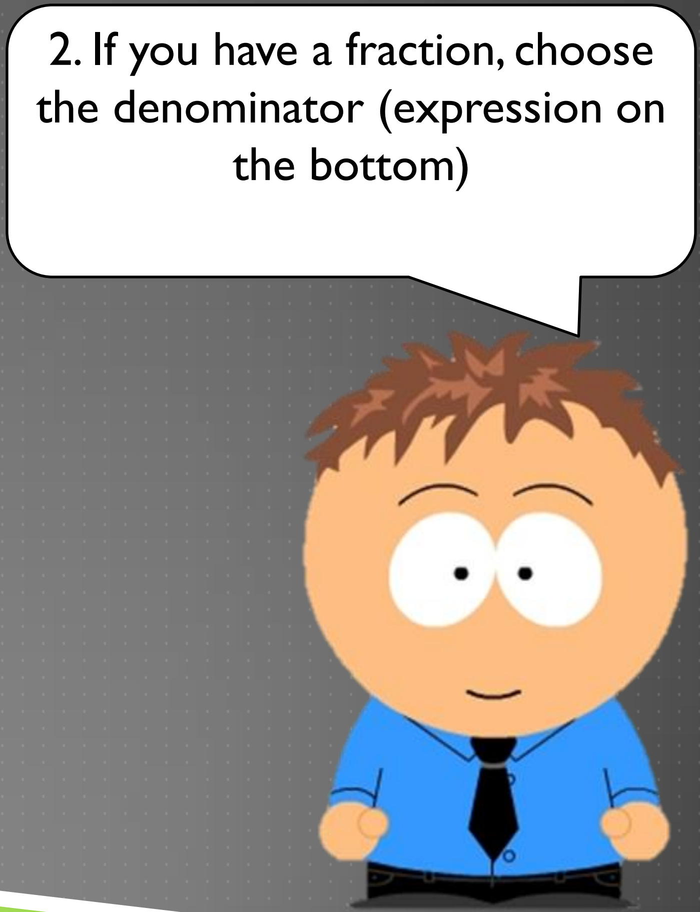

INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

Examples

Expression	**Substitution**
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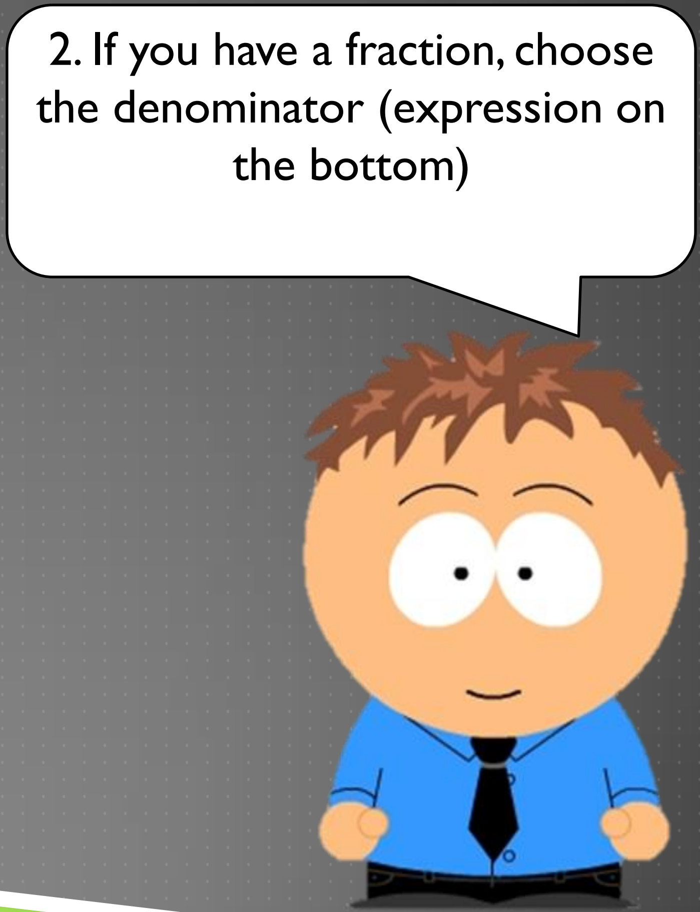
$\int \frac{x^2}{5x^3-1} \; dx$	$u = 5x^3 - 1$
$\int 3xe^{x^2} \; dx$	
$\int \frac{e^x}{\sqrt{e^x+1}} \; dx$	



INTEGRATION BY SUBSTITUTION

Integrating more complex expressions

Examples

Expression	**Substitution**
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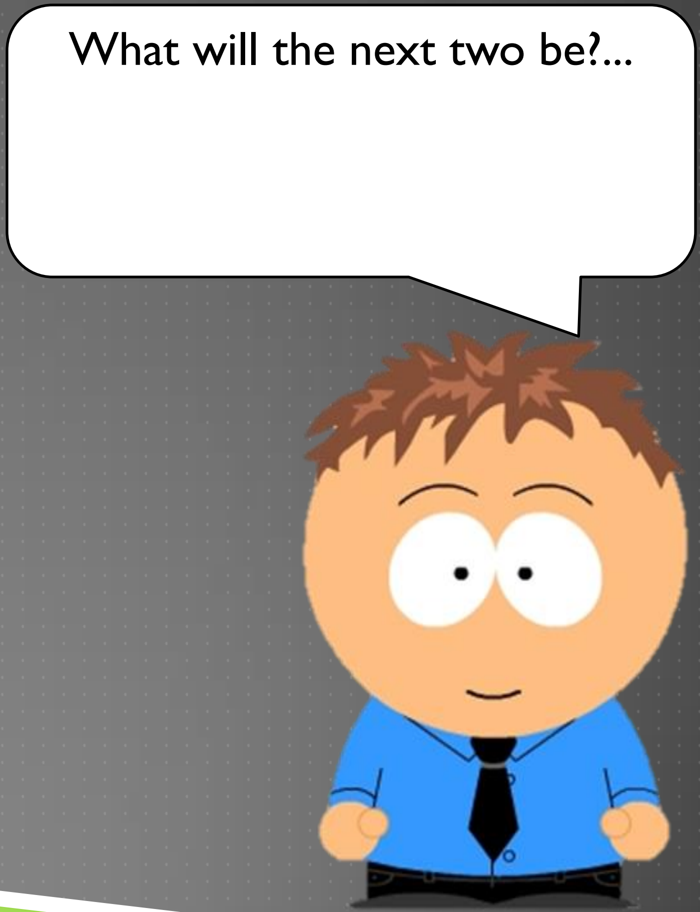

INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

Examples

Expression	**Substitution**
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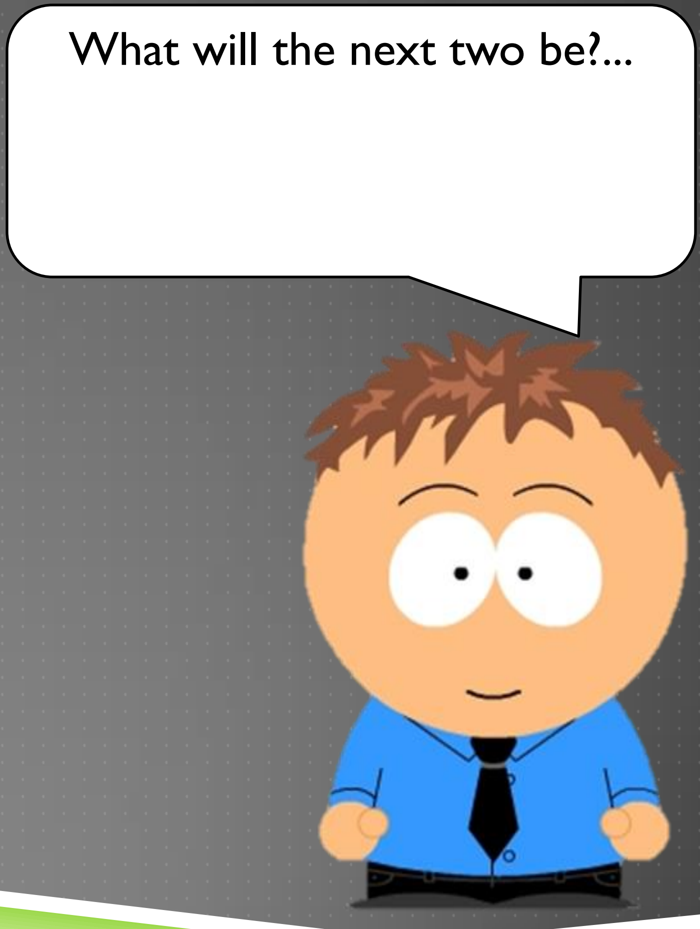

INTEGRATION BY SUBSTITUTION

<u>Integrating more complex expressions</u>

Examples

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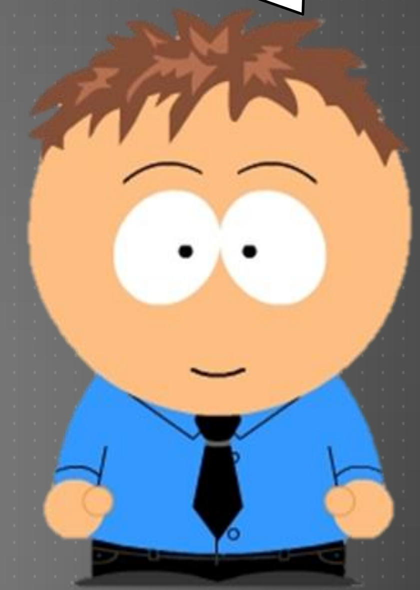
$\int \frac{e^x}{\sqrt{e^x+1}}\ dx$	$u = e^x + 1$

What will the next two be?...



EXAMPLES

1. $$f(x) = \frac{\sin(x)}{\cos(x)}$$

Find $f'(x)$

Find $\int f(x)\,dx$

2. $$g(x) = 5x^2\sqrt{1 - x^3}$$

Find $g'(x)$

Find $\int_0^1 g(x)\,dx$

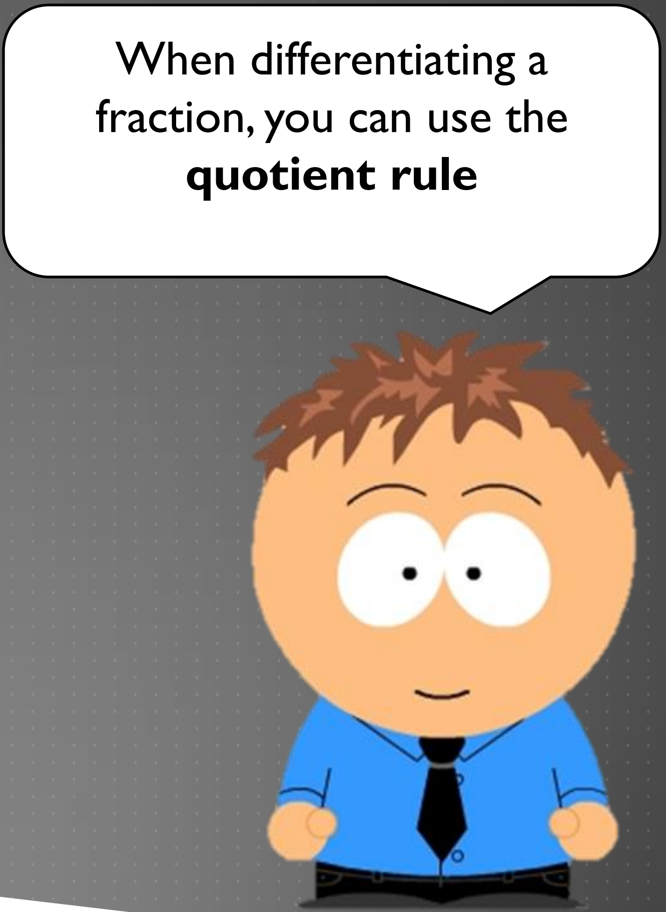
SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $f'(x)$

$$f(x) = \frac{\sin(x)}{\cos(x)}$$



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

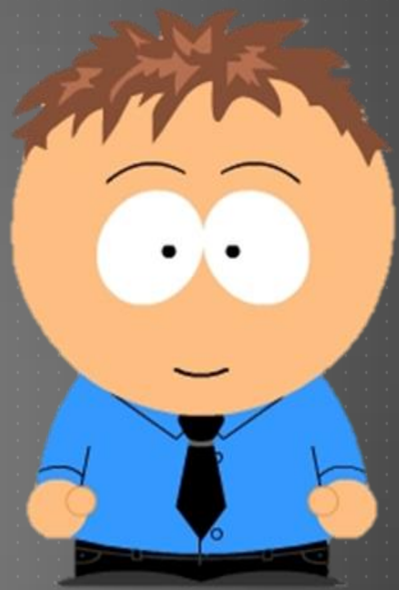
1.

Find $f'(x)$

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

Formula book:

$$y = \frac{u}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

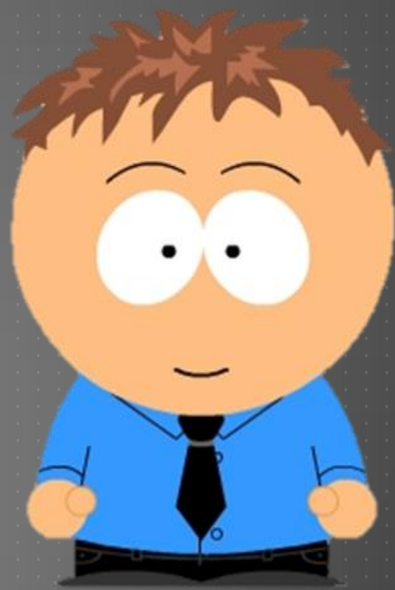
Find $f'(x)$

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

$\sin(x) \leftarrow$ **u**

$\cos(x) \leftarrow$ **v**

Formula book: $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$



SOLUTIONS

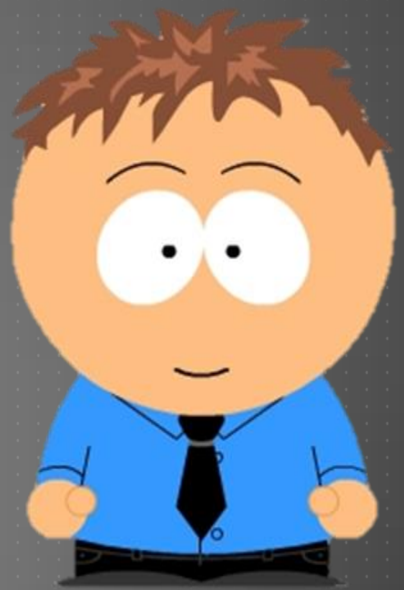
$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $f'(x)$

$$f(x) = \frac{\sin(x)}{\cos(x)} \begin{matrix} \longleftarrow \mathbf{u} \\ \longleftarrow \mathbf{v} \end{matrix}$$

Calculate the derivatives of u and v



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

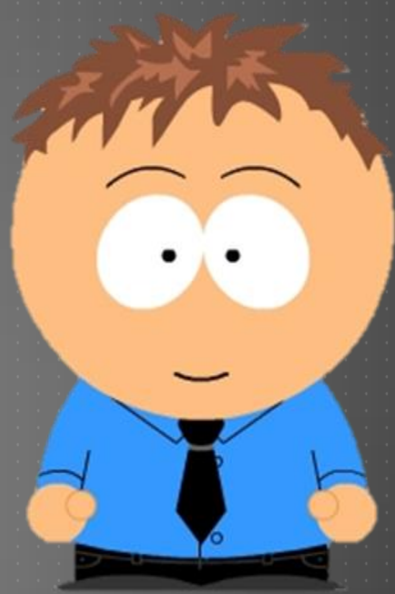
Find $f'(x)$

$$f(x) = \frac{\sin(x) \leftarrow u}{\cos(x) \leftarrow v}$$

$$\frac{du}{dx} = \cos(x)$$

$$\frac{dv}{dx} = -\sin(x)$$

Calculate the derivatives of u and v



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

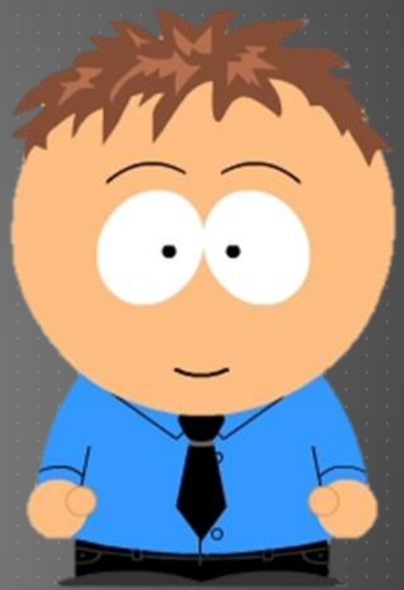
Find $f'(x)$

$$f(x) = \frac{\sin(x) \longleftarrow \mathbf{u}}{\cos(x) \longleftarrow \mathbf{v}}$$

$$\frac{du}{dx} = \cos(x)$$

$$\frac{dv}{dx} = -\sin(\mathrm{x})$$

Substitute all the values back into the formula



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $f'(x)$

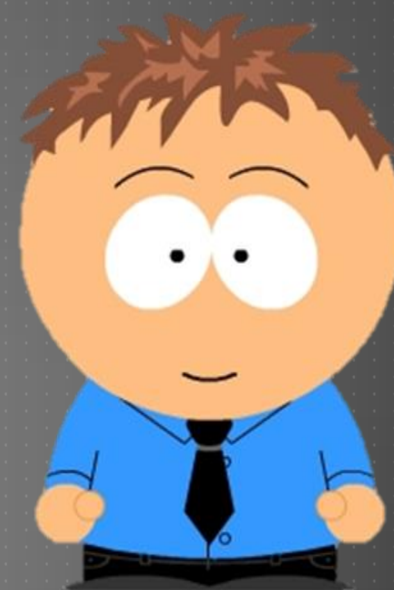
$$f(x) = \frac{\sin(x)}{\cos(x)} \begin{matrix} \leftarrow \mathbf{u} \\ \leftarrow \mathbf{v} \end{matrix}$$

$$\frac{du}{dx} = \cos(x)$$

$$\frac{dv}{dx} = -\sin(\mathrm{x})$$

$$f'(x) = \frac{\cos(x).\cos(x) - \sin(x).\sin(x)}{(\cos(x))^2}$$

Formula book: $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $f'(x)$

$$f(x) = \frac{\sin(x)}{\cos(x)} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix} \qquad \frac{du}{dx} = \cos(x)$$

$$\frac{dv}{dx} = -\sin(\mathrm{x})$$

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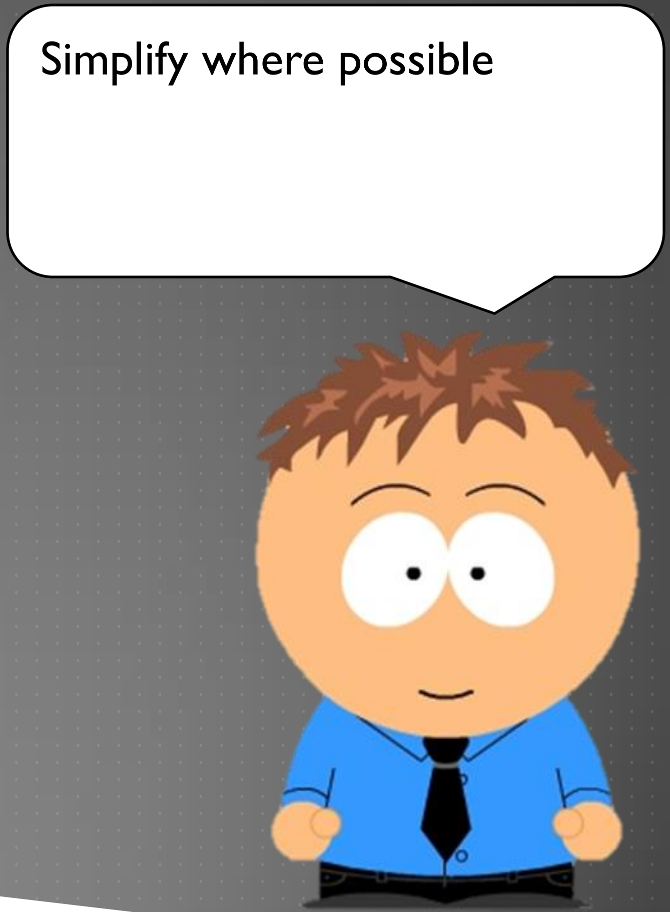


SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

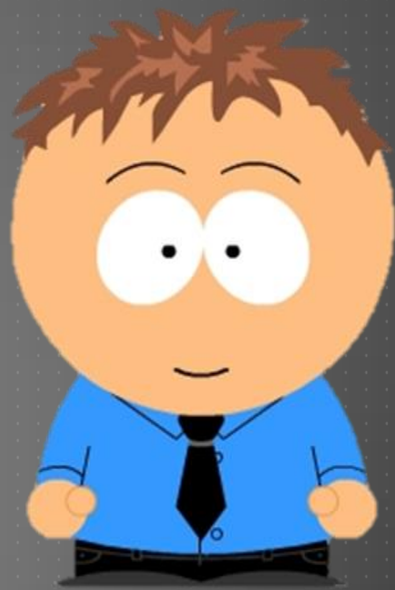
Find $f'(x)$

$$f(x) = \frac{\sin(x)}{\cos(x)} \quad \begin{matrix}\leftarrow u \\ \leftarrow v\end{matrix} \qquad \frac{du}{dx} = \cos(x)$$

$$\frac{dv}{dx} = -\sin(x)$$

$$f'(x) = \frac{\cos(x).\cos(x) - \sin(x).\sin(x)}{(\cos(x))^2}$$

Remember that
$\cos(x) \times \cos(x) = \cos^2 x$



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $f'(x)$

$$f(x) = \frac{\sin(x) \leftarrow u}{\cos(x) \leftarrow v}$$

$$\frac{du}{dx} = \cos(x)$$

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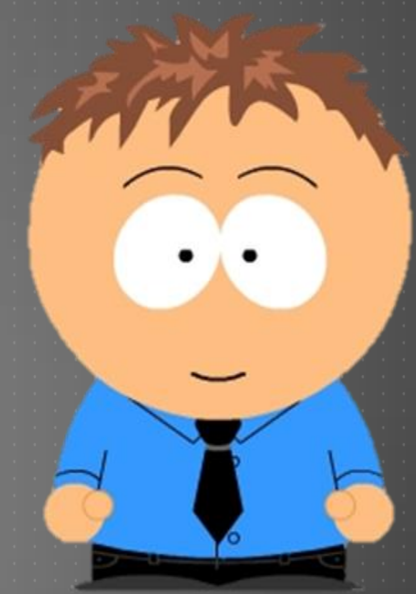
$$f'(x) = \frac{\cos(x).\cos(x) - \sin(x).-\sin(x)}{(\cos(x))^2}$$

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Remember that

$$\cos(x) \times \cos(x) = \cos^2 x$$



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $f'(x)$

$$f(x) = \frac{\sin(x) \longleftarrow \mathbf{u}}{\cos(x) \longleftarrow \mathbf{v}} \qquad \frac{du}{dx} = \cos(x)$$

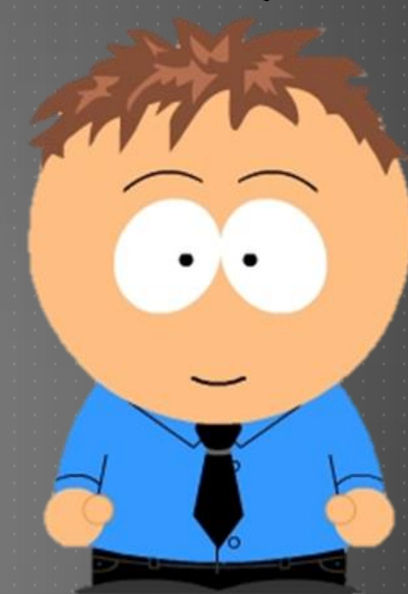
$$\frac{dv}{dx} = -\sin(x)$$

$$f'(x) = \frac{\cos(x).\cos(x) - \sin(x)\,.-\sin(x)}{(\cos(x))^2}$$

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad = \frac{\mathbf{1}}{\mathbf{\cos^2 x}}$$

And that $\sin^2$x and $\cos^2$x are related by:

$$\sin^2 x + \cos^2 x = 1$$



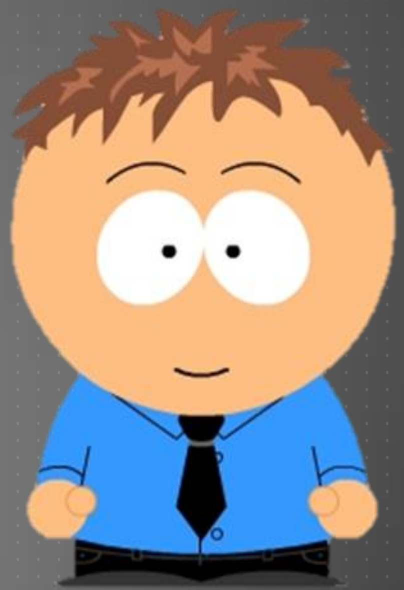
SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $\int f(x)\, dx$

$$\int \frac{\sin(x)}{\cos(x)} dx$$



SOLUTIONS

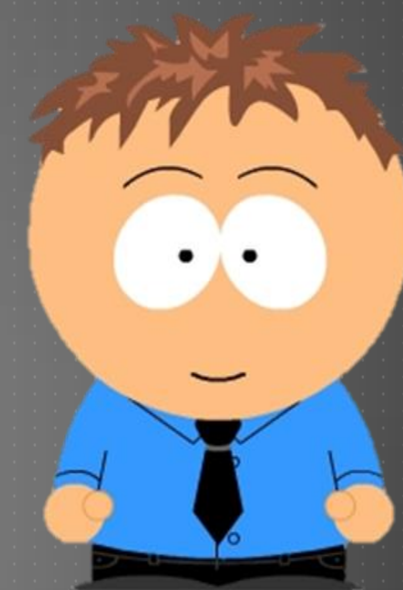
$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $\int f(x)\, dx$

$$\int \frac{\sin(x)}{\cos(x)} dx$$

Choose a suitable substitution and differentiate to find how to change **dx** to **du**



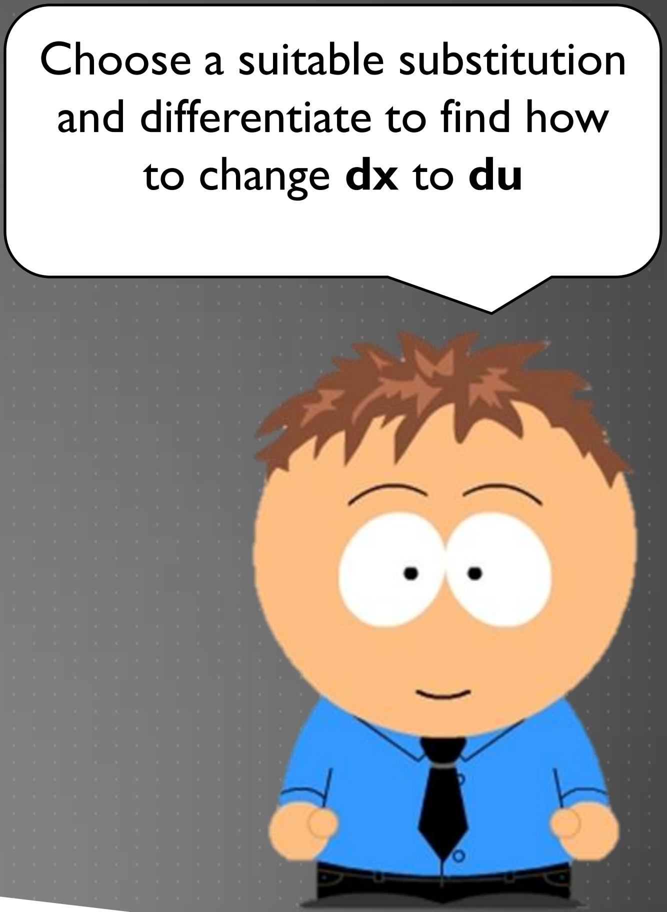
SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1\.

Find $\int f(x)\,dx$

$$\int \frac{\sin(x)}{\cos(x)}\,dx \qquad u = \cos(x)$$



SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

1.

Find $\int f(x)\,dx$

$$\int \frac{\sin(x)}{\cos(x)}\,dx$$

$$u = \cos(x)$$

$$\frac{du}{\boldsymbol{dx}} = -\sin(x)$$

SOLUTIONS

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

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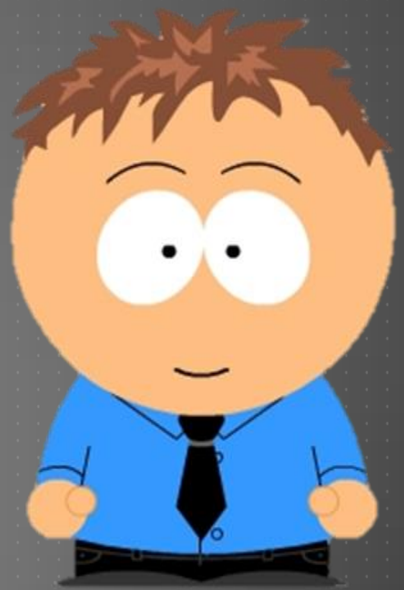

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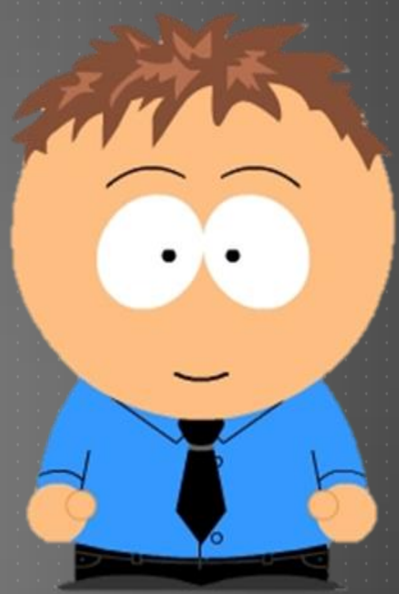
Rewrite the integral



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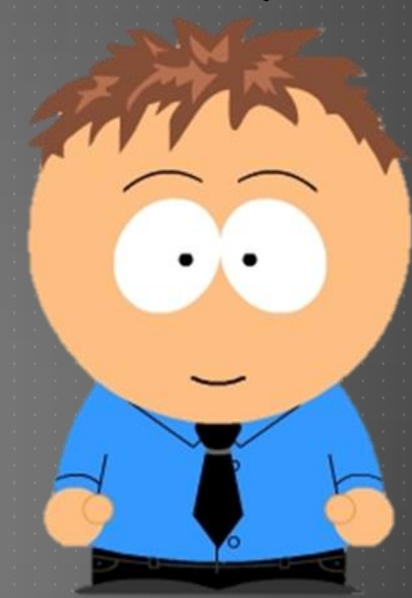
$$\int \frac{\sin(x)}{\cos(x)}dx \qquad = \int \frac{\sin(x)}{u}\frac{du}{-\sin(x)}$$

Rewrite the integral



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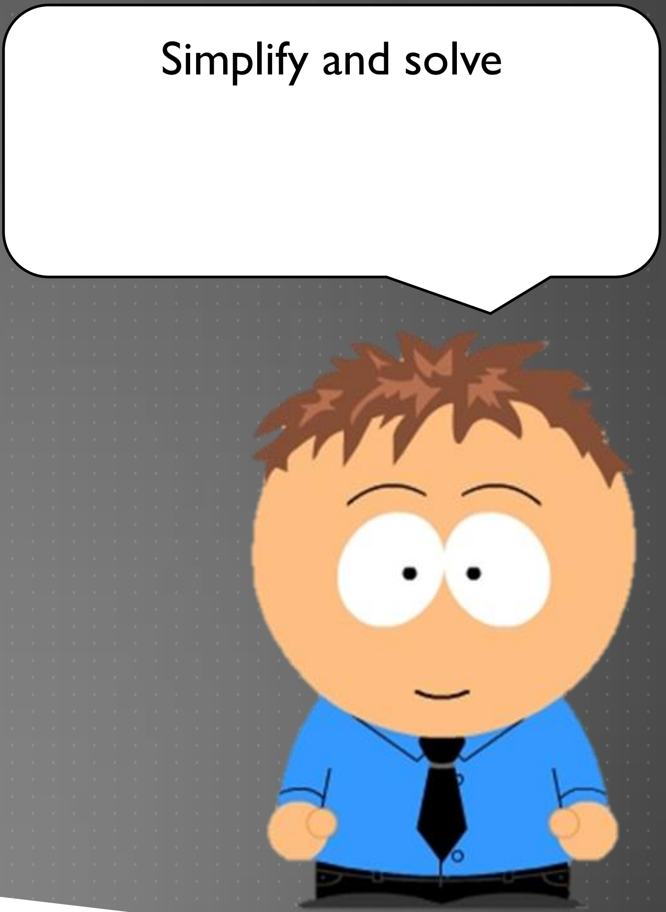

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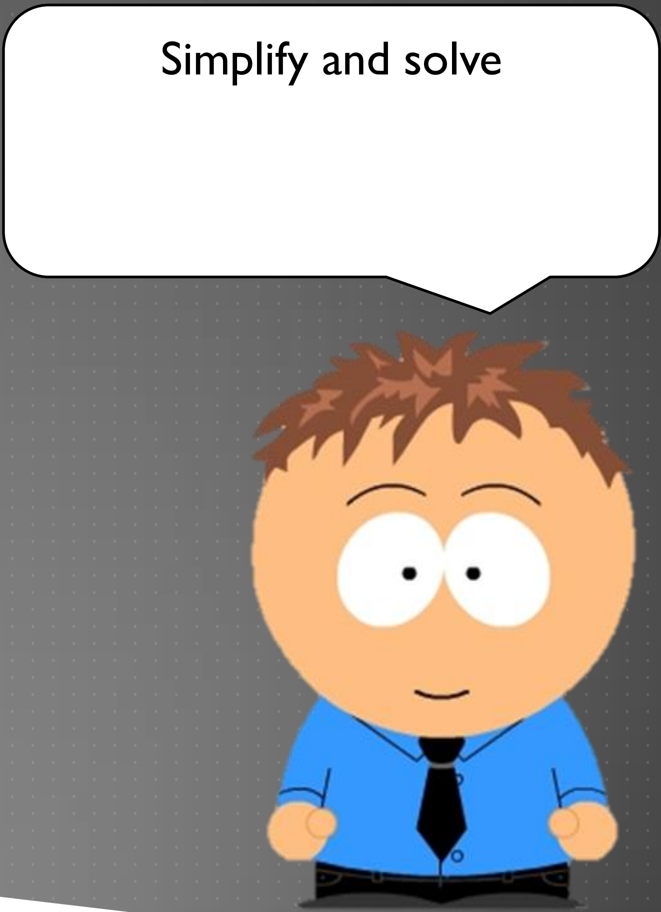
$$= \int -\frac{1}{u} du$$



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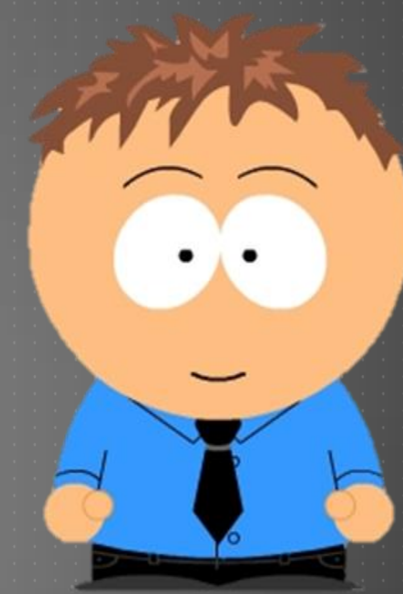
$$= \int -\frac{1}{u}du = -\ln(u) + C$$



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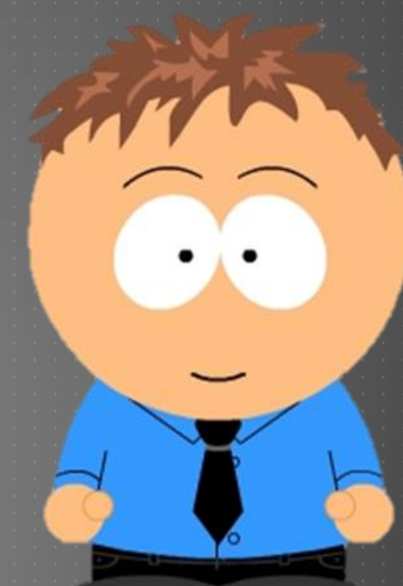
Substitute x back in for u



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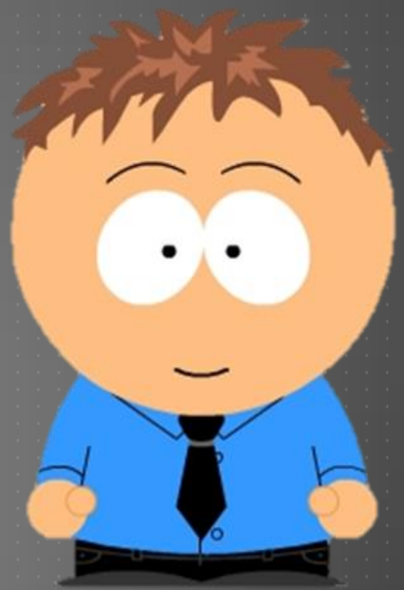
$$= \boldsymbol{-\ln(\cos(x)) + C}$$



EXAMPLES

1. $$f(x) = \frac{\sin(x)}{\cos(x)}$$

Find $f'(x)$

Find $\int f(x)\,dx$

2. $$g(x) = 5x^2\sqrt{1 - x^3}$$

Find $g'(x)$

Find $\int_0^1 g(x)\,dx$

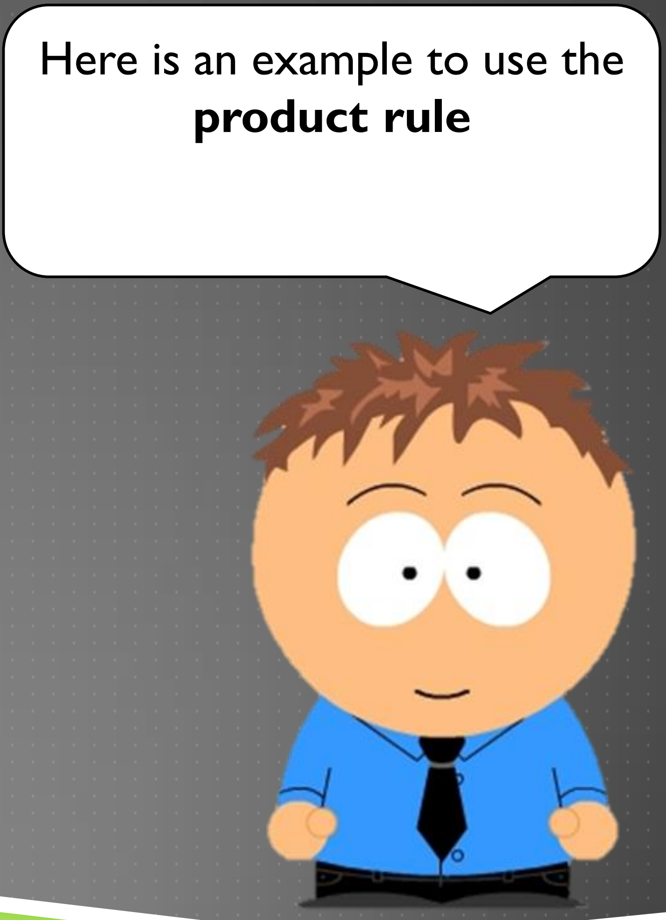
SOLUTIONS

$$g(x) = 5x^2\sqrt{1 - x^3}$$

2.

Find $g'(x)$

$g(x) = 5x^2\sqrt{1 - x^3}$



SOLUTIONS

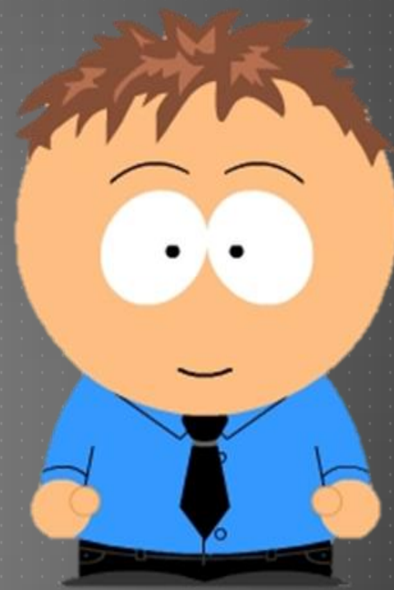
$$g(x) = 5x^2\sqrt{1 - x^3}$$

2.

Find $g'(x)$

$$g(x) = 5x^2\sqrt{1 - x^3}$$

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



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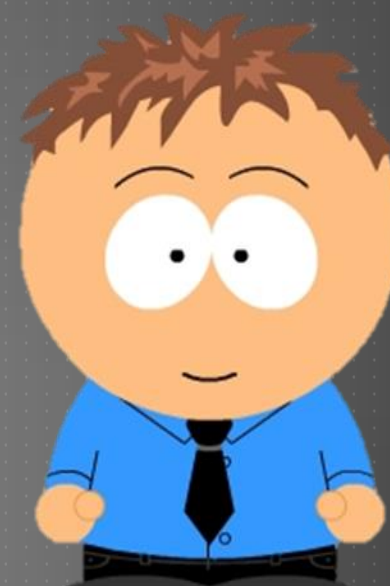

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2.

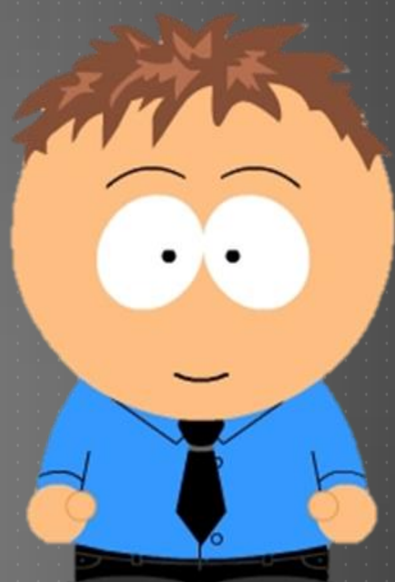
Find $g'(x)$

$$g(x) = 5x^2\sqrt{1 - x^3}$$

$u = 5x^2 \qquad v = \sqrt{1 - x^3}$

$$\frac{du}{dx} = 10x$$

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



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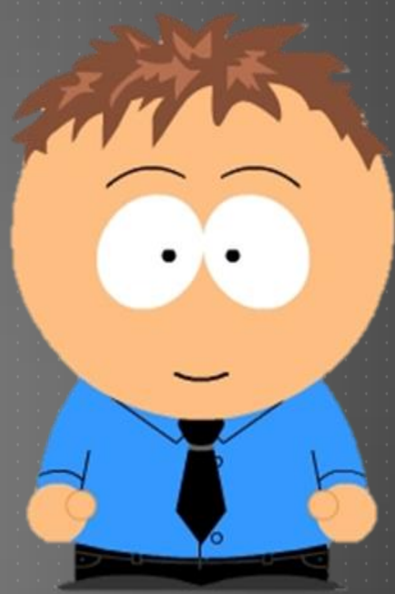
Find $g'(x)$

$$g(x) = 5x^2\sqrt{1 - x^3}$$

$$u = 5x^2 \qquad v = \sqrt{1 - x^3} = (1 - x^3)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 10x \qquad \frac{dv}{dx} = -3x^2 \times \frac{1}{2}(1 - x^3)^{-\frac{1}{2}}$$

You might want to rewrite v using powers, to more easily differentiate using the chain rule



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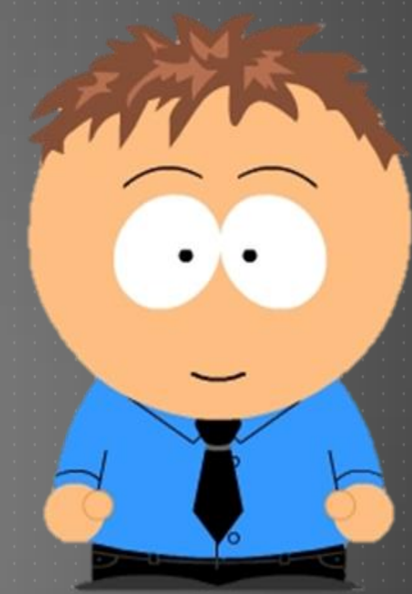
$$g'(x) =$$

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



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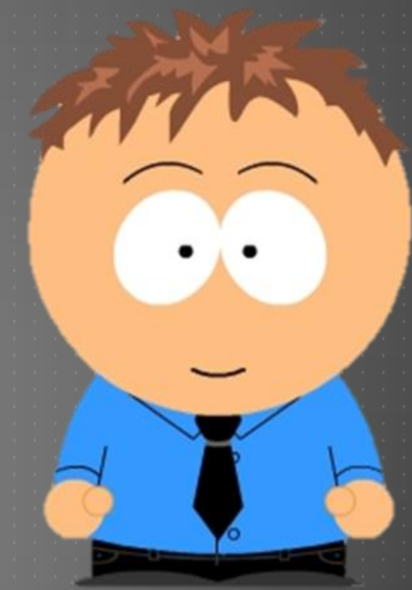
$$g'(x) = 5x^2 \times -\frac{3x^2}{2\sqrt{1-x^3}}$$

$$y = uv \quad \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



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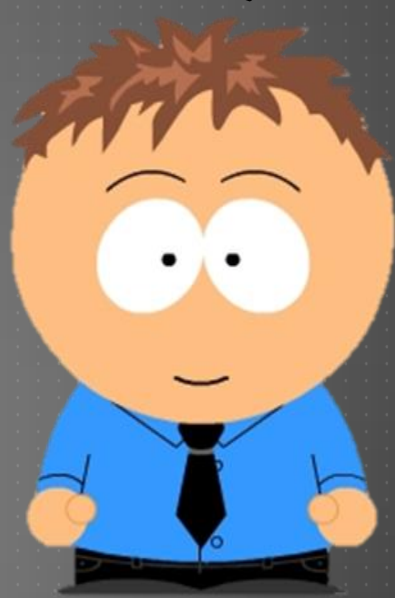
$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



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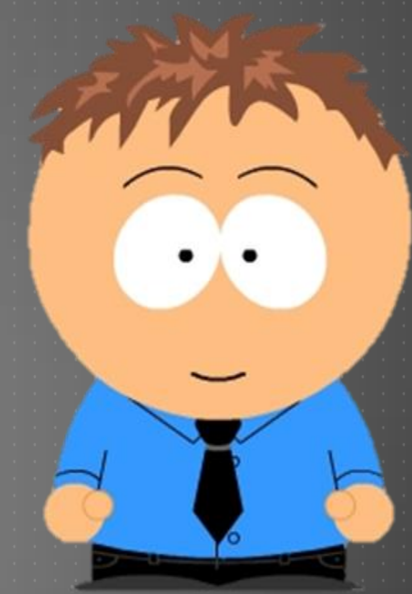
$$g'(x) = 5x^2 \times -\frac{3x^2}{2\sqrt{1-x^3}} + \sqrt{1-x^3} \times 10x$$

This can be further simplified. It is a little bit tricky, but by this stage you will have gained the majority of the marks



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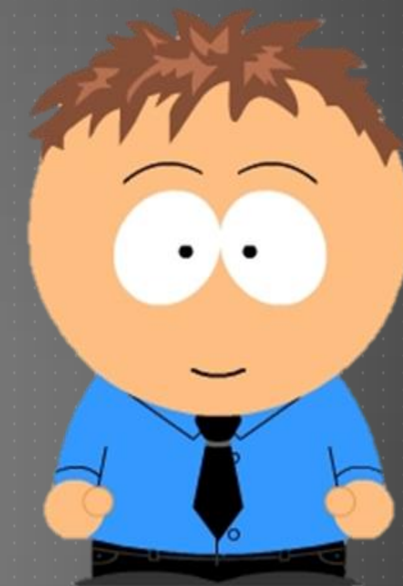
$$g'(x) = 5x^2 \times -\frac{3x^2}{2\sqrt{1-x^3}} + \sqrt{1-x^3} \times 10x$$

$$= -\frac{15x^4}{2\sqrt{1-x^3}} + \frac{20x(1-x^3)}{2\sqrt{1-x^3}}$$

Writing over common denominator helps



SOLUTIONS

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$$g(x) = 5x^2\sqrt{1-x^3}$$

Find $g'(x)$

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$$g'(x) = 5x^2 \times -\frac{3x^2}{2\sqrt{1-x^3}} + \sqrt{1-x^3} \times 10x$$

$$= -\frac{15x^4}{2\sqrt{1-x^3}} + \frac{20x(1-x^3)}{2\sqrt{1-x^3}} = \frac{5x(4-7x^3)}{2\sqrt{1-x^3}}$$

And the final result (with a few steps skipped), can be seen below

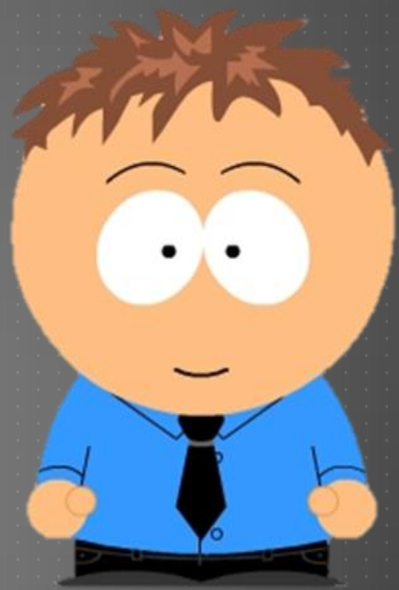
SOLUTIONS

$$g(x) = 5x^2\sqrt{1 - x^3}$$

2.

Find $\int_0^1 g(x)\,dx$

$$\int_0^1 5x^2\sqrt{1 - x^3}\,dx$$



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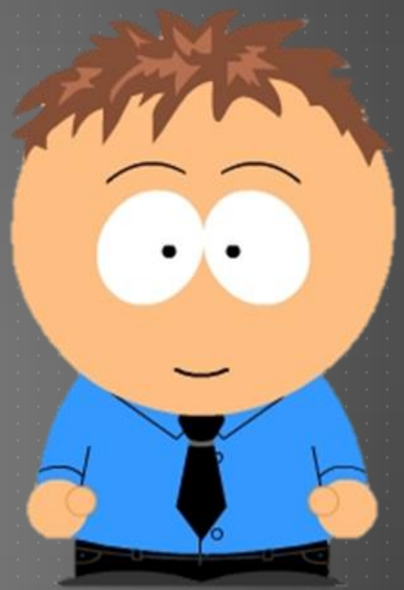

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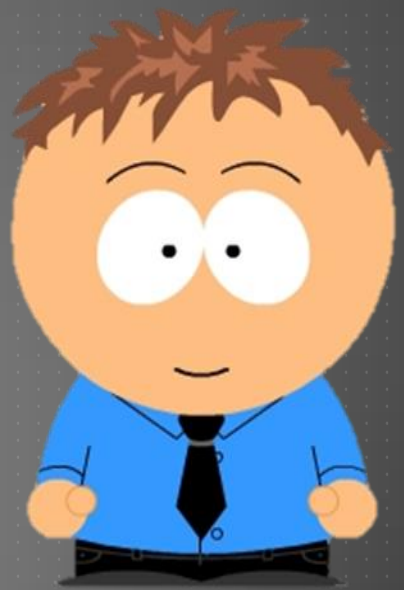

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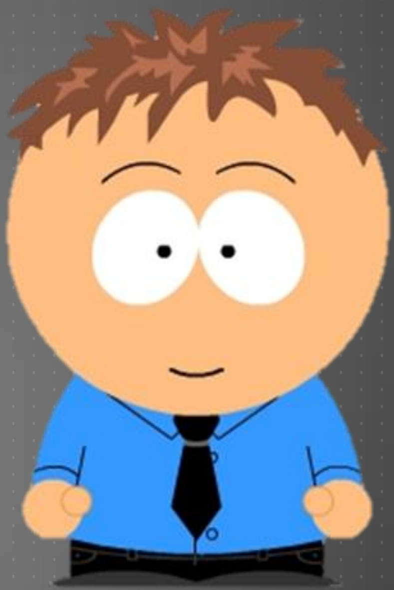

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$$\int 5x^2 u^{\frac{1}{2}}\ \frac{du}{-3x^2}$$

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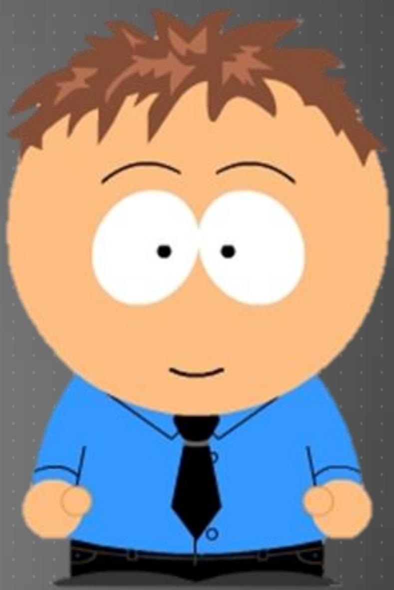

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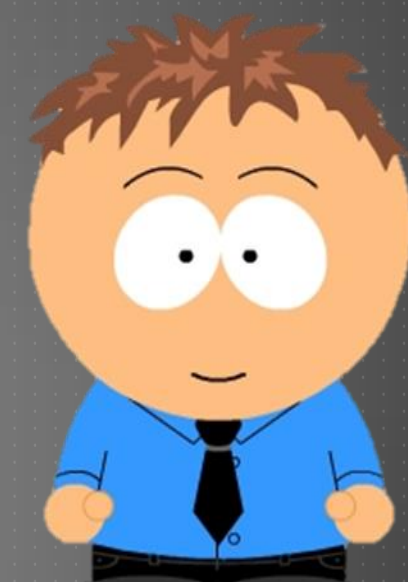
$$\int 5x^2 u^{\frac{1}{2}} \frac{du}{-3x^2} = \int -\frac{5}{3}u^{\frac{1}{2}}\, du = -\frac{10}{9}u^{\frac{3}{2}}$$

Remember that when integrating between limits we **don't need to include '+c'**



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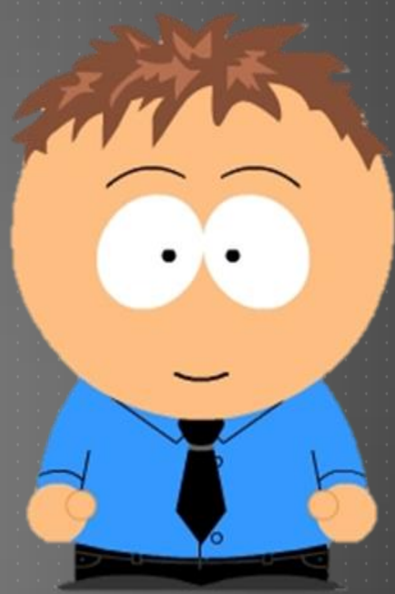
$$\left[-\frac{10}{9}(1 - x^3)^{\frac{3}{2}}\right]_0^1$$



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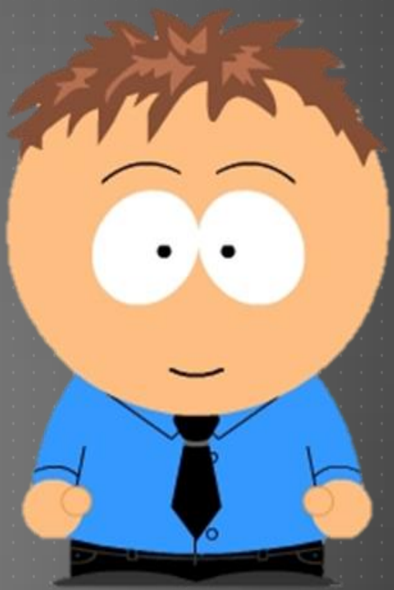
$$\left[-\frac{10}{9}(1-x^3)^{\frac{3}{2}}\right]_0^1 = 0 - -\frac{10}{9}$$



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