

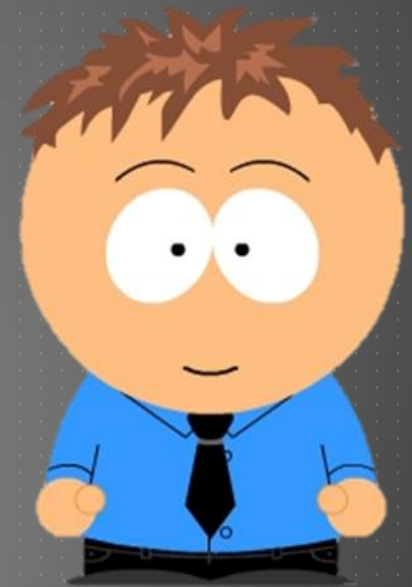
# DIFFERENTIATION AND INTEGRATION PART 2

Mr C's IB Standard Notes

In this PDF you can find the following:

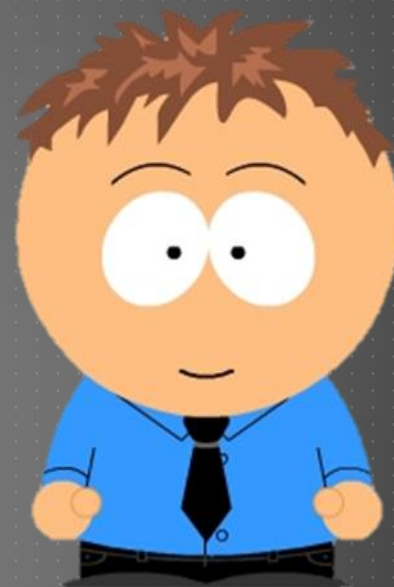
1. [The Chain Rule](#)
2. [Integration by Inspection](#)
3. [Worked Examples](#)

Make sure you read through everything and then try examples for yourself before looking at the solutions



# THE CHAIN RULE

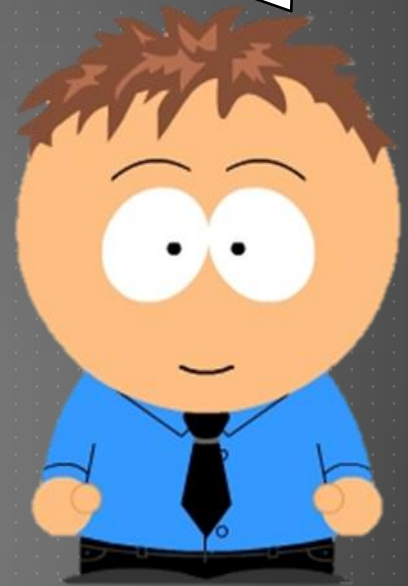
Differentiating Composite Functions



# THE CHAIN RULE

## Differentiating Composite Functions

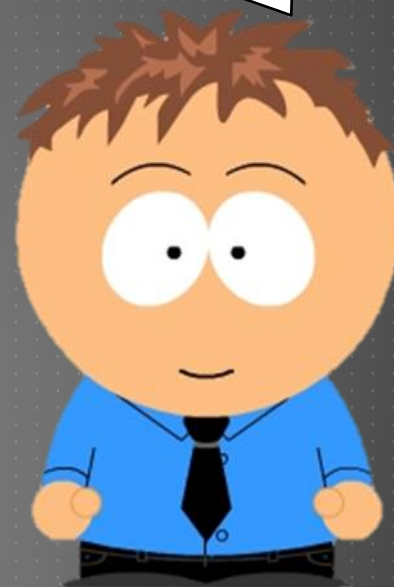
Whilst it's easy to differentiate functions like  $x^5$ ,  $(x + 2)^5$  would take a very long time (if multiplying out brackets first)



# THE CHAIN RULE

## Differentiating Composite Functions

Fortunately the *chain rule* makes these types of derivatives quick and easy to find

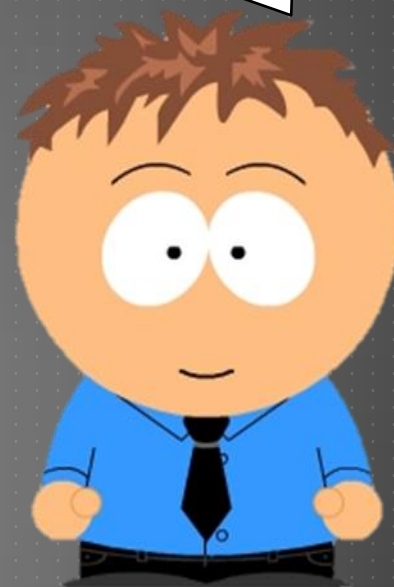


# THE CHAIN RULE

## Differentiating Composite Functions

E.g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

Fortunately the *chain rule* makes these types of derivatives quick and easy to find

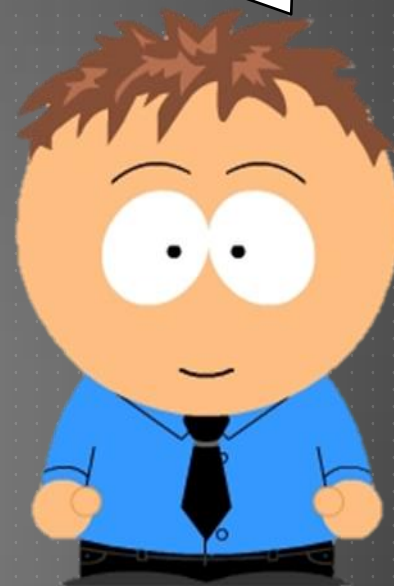


# THE CHAIN RULE

## Differentiating Composite Functions

E.g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

These are all functions of functions, or *composite functions*



# THE CHAIN RULE

## Differentiating Composite Functions

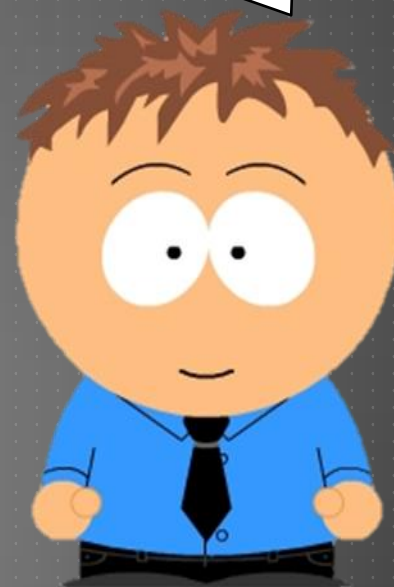
E.g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

E.g.

$$f(x) = 2x + 1$$

$$g(x) = x^5$$

$$gf(x) = (2x + 1)^5$$



# THE CHAIN RULE

## Differentiating Composite Functions

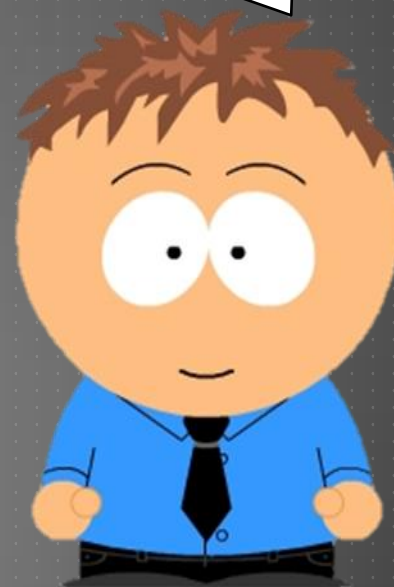
E.g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

E.g.

$$f(x) = 3x$$

$$g(x) = \sin(x)$$

$$gf(x) = \sin(3x)$$

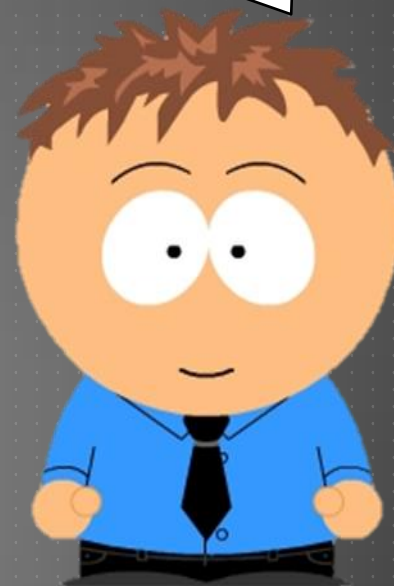


# THE CHAIN RULE

## Differentiating Composite Functions

E.g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

E.g.  
 $f(x) = x^3$   
 $g(x) = e^x$   
 $gf(x) = e^{x^3}$

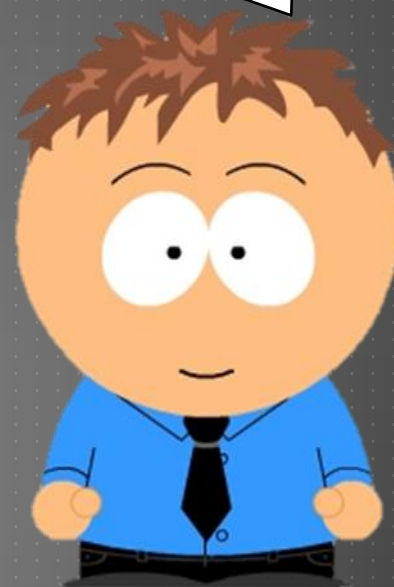


# THE CHAIN RULE

## Differentiating Composite Functions

*E.g.*  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

To differentiate you follow a  
simple rule

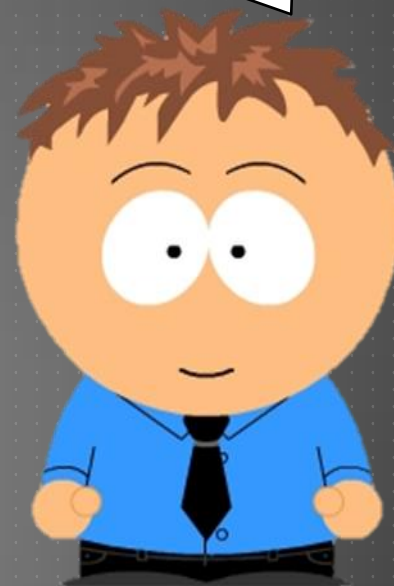


# THE CHAIN RULE

## Differentiating Composite Functions

E.g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

1. Differentiate the *inner* function,  
 $f(x)$



# THE CHAIN RULE

## Differentiating Composite Functions

E. g.  $(2x + 1)^5$ ,



$\sin(3x)$ ,



$e^{x^3}$



1. Differentiate the *inner* function,  
 $f(x)$



# THE CHAIN RULE

## Differentiating Composite Functions

E. g.  $(2x + 1)^5$ ,

$\sin(3x)$ ,

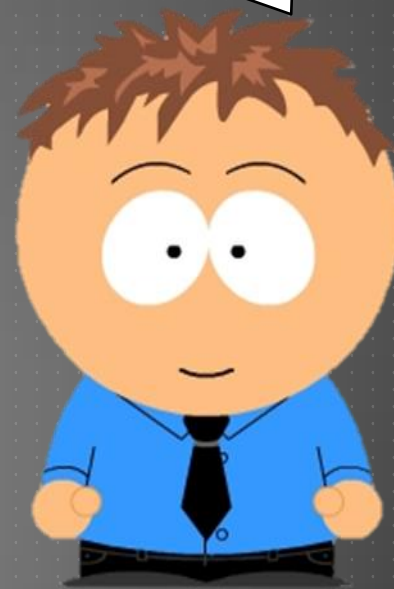
$e^{x^3}$

$$\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = 3x^2$$

I. Differentiate the *inner* function,  
 $f(x)$

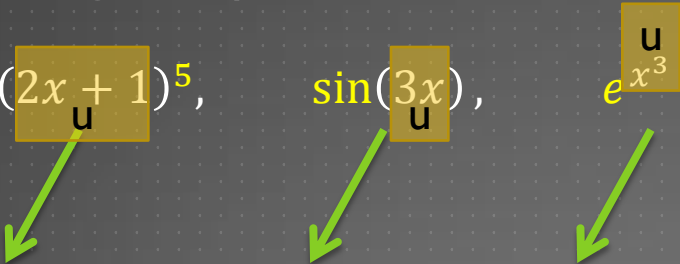


# THE CHAIN RULE

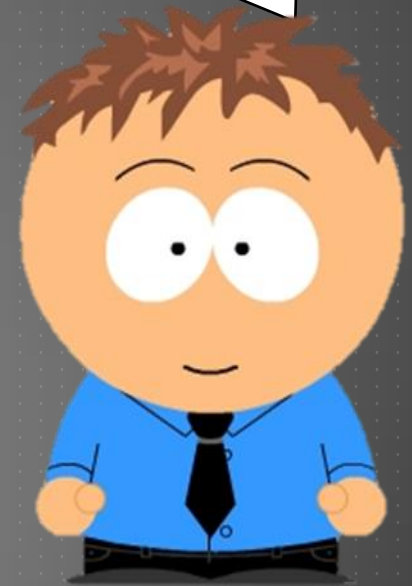
## Differentiating Composite Functions

E. g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

$\frac{dy}{dx} = 2$        $\frac{dy}{dx} = 3$        $\frac{dy}{dx} = 3x^2$



2. Replace the inner function with a single variable, e.g.  $u$ , and differentiate



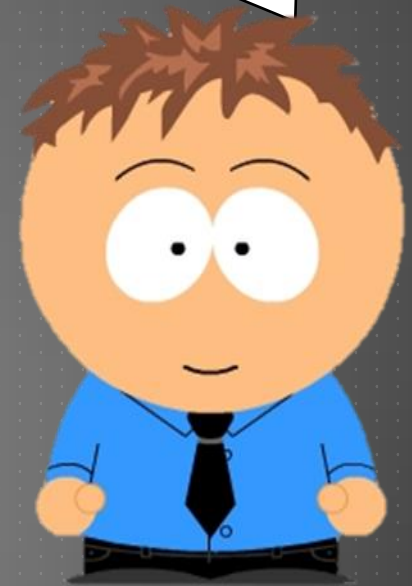
# THE CHAIN RULE

## Differentiating Composite Functions

E. g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

$\frac{dy}{dx} = 2 \cdot 5u^4$        $\frac{dy}{dx} = 3 \cos(u)$        $\frac{dy}{dx} = 3x^2 \cdot e^u$

2. Replace the inner function with a single variable, e.g.  $u$ , and differentiate



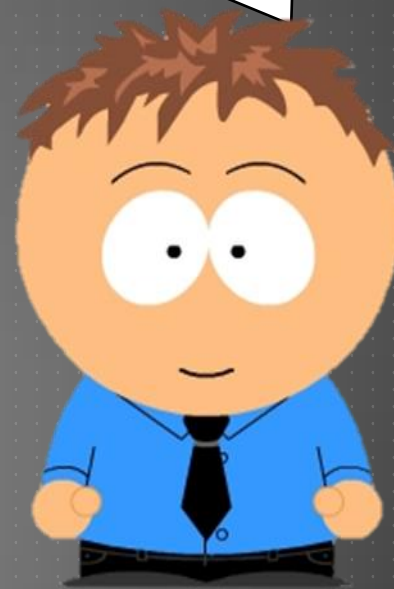
# THE CHAIN RULE

## Differentiating Composite Functions

E. g.  $(2x + 1)^5$ ,  $\sin(3x)$ ,  $e^{x^3}$

$\frac{dy}{dx} = 2 \cdot 5u^4$        $\frac{dy}{dx} = 3 \cos(u)$        $\frac{dy}{dx} = 3x^2 \cdot e^u$

3. Multiply the results and replace u



# THE CHAIN RULE

## Differentiating Composite Functions

E. g.  $(2x + 1)^5$ ,  
          u



$$\frac{dy}{dx} = 2 \cdot 5u^4$$

$$= 10(2x + 1)^4$$

$\sin(3x)$ ,  
          u



$$\frac{dy}{dx} = 3\cos(u)$$

$$= 3\cos(3x)$$

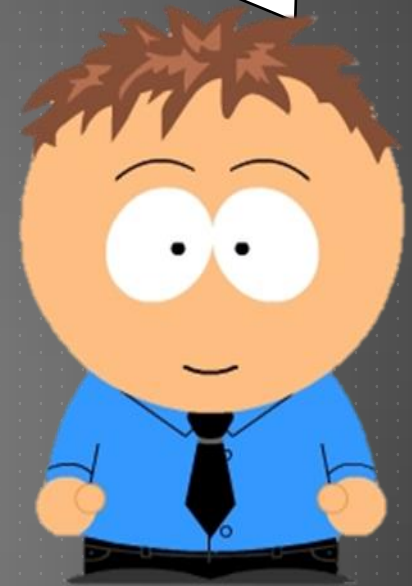
$e^{x^3}$   
          u



$$\frac{dy}{dx} = 3x^2 e^u$$

$$= 3x^2 e^{x^3}$$

3. Multiply the results and  
replace u



# THE CHAIN RULE

## Differentiating Composite Functions

E.g.  $(2x + 1)^5$ ,

$\sin(3x)$ ,

$e^{x^3}$



$$\frac{dy}{dx} = 10(2x + 1)^4$$

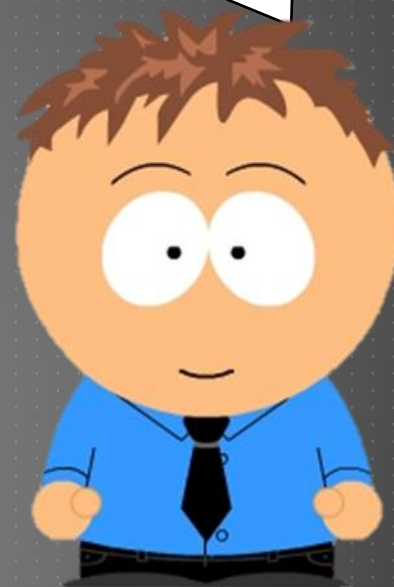


$$\frac{dy}{dx} = 3\cos(3x)$$



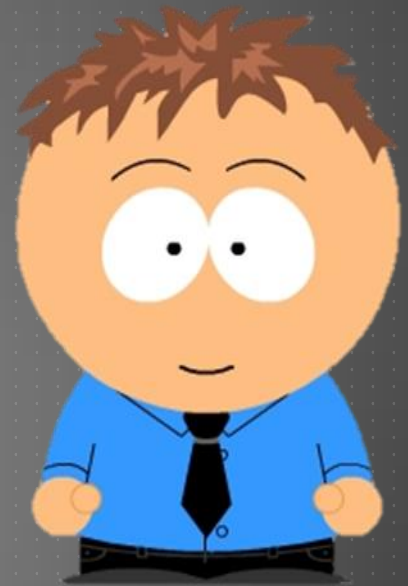
$$\frac{dy}{dx} = 3x^2 e^{x^3}$$

With practice you probably won't need the middle step, and can instead jump straight to the final solution



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

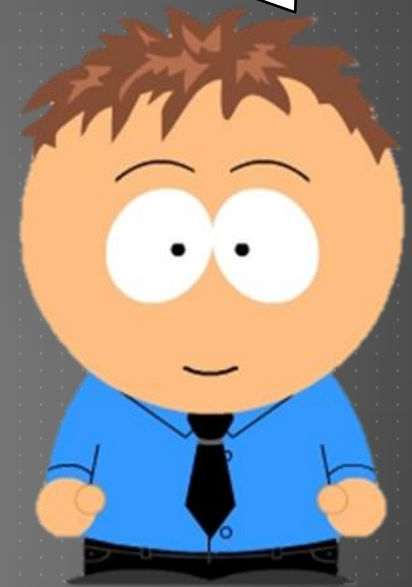


# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

E.g. Evaluate  $\int (2x - 3)^4 dx$

There is no exact rule for integration, but sometimes you can try to imagine working backwards to solve integrals



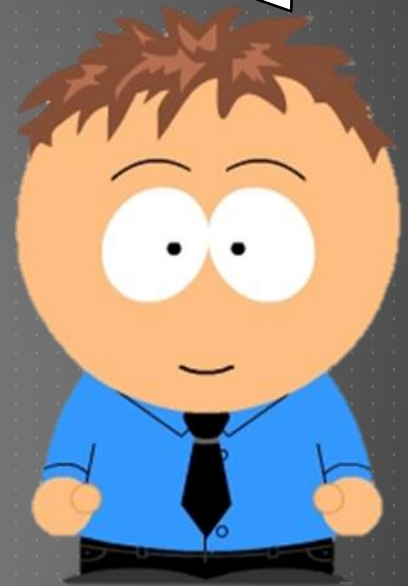
# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

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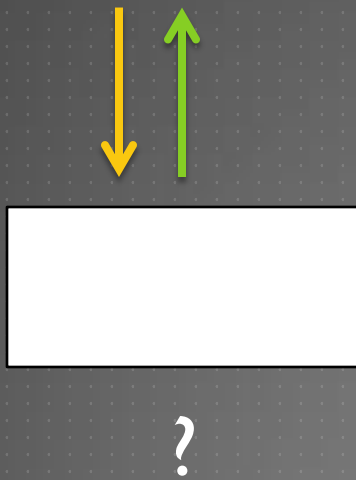
Try to think about what you would have to differentiate to get the result in the question



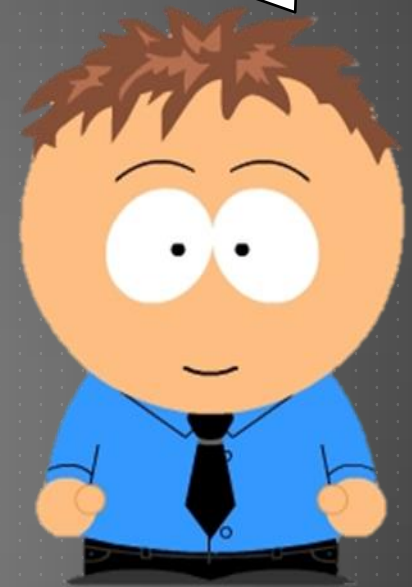
# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

E.g. Evaluate  $\int (2x - 3)^4 dx$



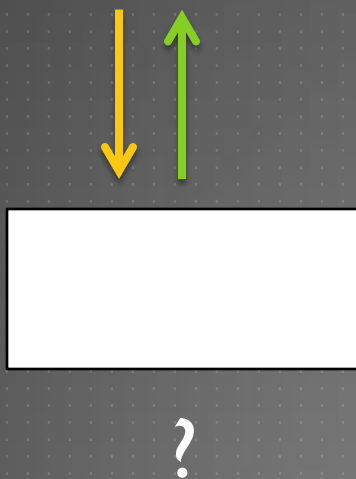
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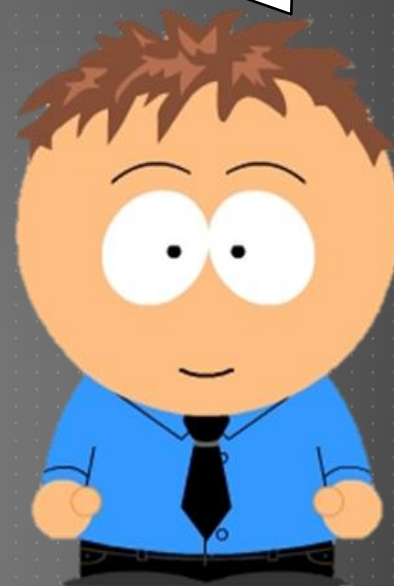
# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

E.g. Evaluate  $\int (2x - 3)^4 dx$




We know to end up with  
 $(2x - 3)^4$   
we need something like  
 $(2x - 3)^5$



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

E.g. Evaluate  $\int (2x - 3)^4 dx$



$$(2x - 3)^5$$

?


We know to end up with  
 $(2x - 3)^4$   
we need something like  
 $(2x - 3)^5$



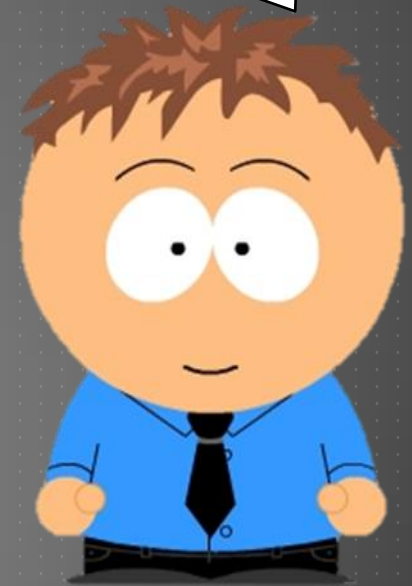
# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

E.g. Evaluate  $\int (2x - 3)^4 dx$


$$\begin{array}{c} 2(2x - 3)^4 \\ \downarrow \uparrow \\ (2x - 3)^5 \\ ? \end{array}$$


When we differentiate  
 $(2x - 3)^5$   
We get  
 $2(2x - 3)^4$



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

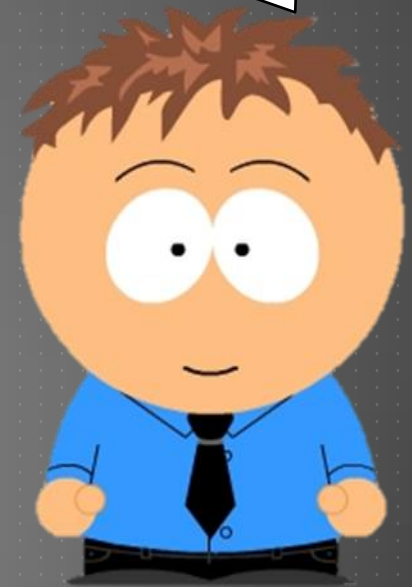
E.g. Evaluate  $\int (2x - 3)^4 dx$

  $2(2x - 3)^4$

$(2x - 3)^5$

?

So we just have to make a small change to get the answer we want.



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

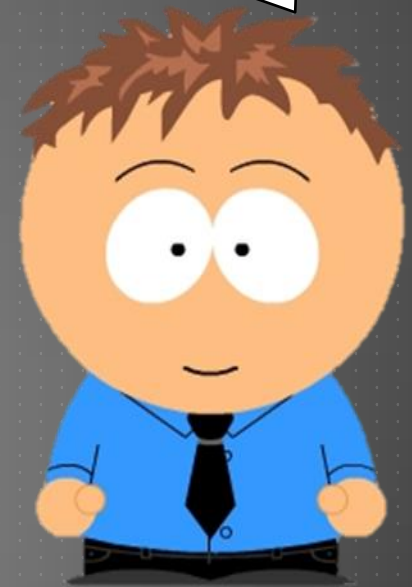
E.g. Evaluate  $\int (2x - 3)^4 dx$

$$\begin{array}{c} \downarrow \uparrow \\ \frac{1}{2} \cdot 2(2x - 3)^4 \end{array}$$

$$\frac{1}{2}(2x - 3)^5$$

?

So we just have to make a small change to get the answer we want.



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

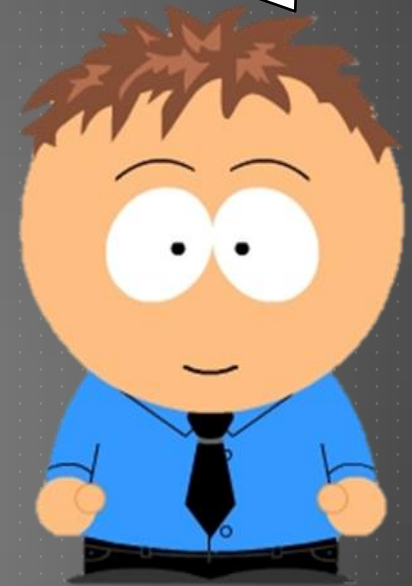
E.g. Evaluate  $\int (2x - 3)^4 dx$

$$\begin{array}{c} \downarrow \uparrow \\ \frac{1}{2} \cdot 2(2x - 3)^4 \end{array}$$

$$\frac{1}{2}(2x - 3)^5 + C$$

?

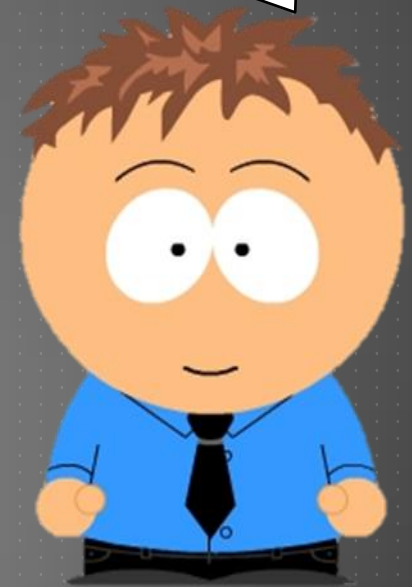
And don't forget to include your  
*constant of integration* at the end!



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

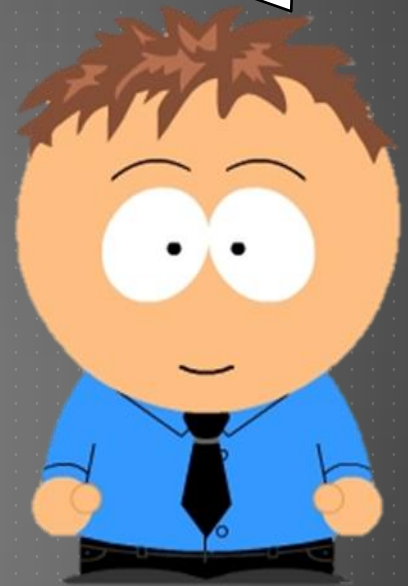
Inspection is a handy method if you get the hang of it, but it's not always easy to visualise.



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

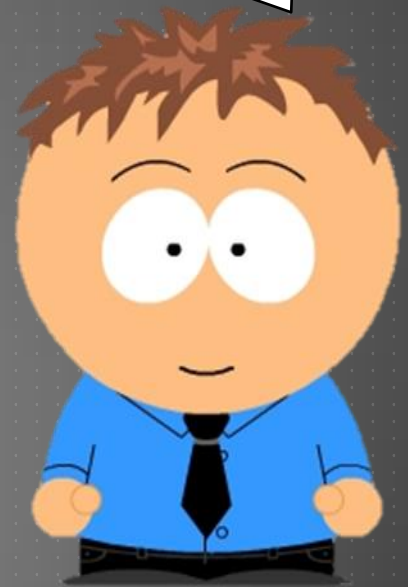
Another method which **always** works in this type of situation is *integration by substitution*



# INTEGRATION BY INSPECTION

Reversing the chain rule for integration

I'll recap that method next time,  
for now why not try a couple  
quick examples:



# EXAMPLES

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

1.

Find the area under the curve  $y = \frac{1}{2}e^{4x}$ , bounded by the  $x$ -axis, and the lines  $x=0$  and  $x=1$

2.

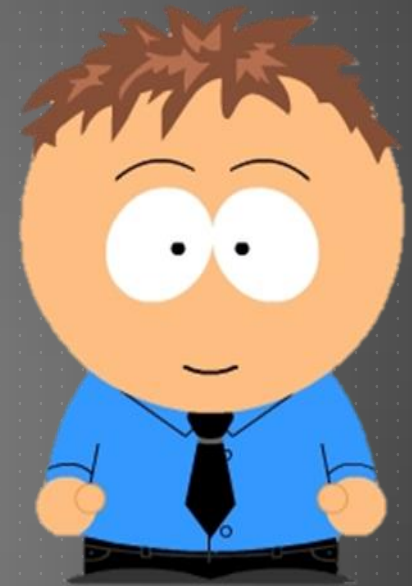
# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

We need to find gradient,  
which means we must  
differentiate



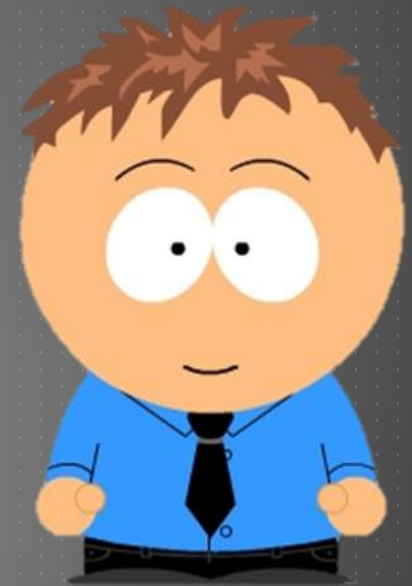
# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

This is a composite function,  
so use the chain rule



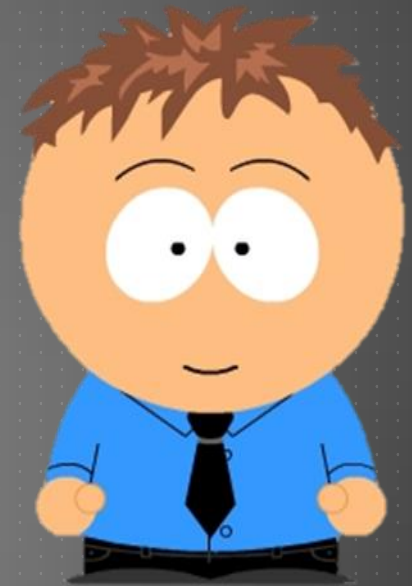
# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

Differentiate the inner function



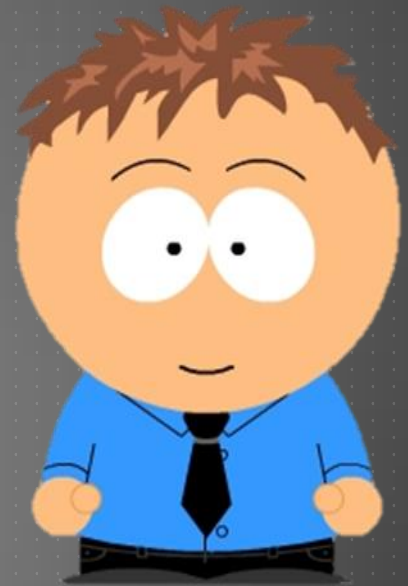
# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

$$f'(x) = 2x$$



# SOLUTIONS

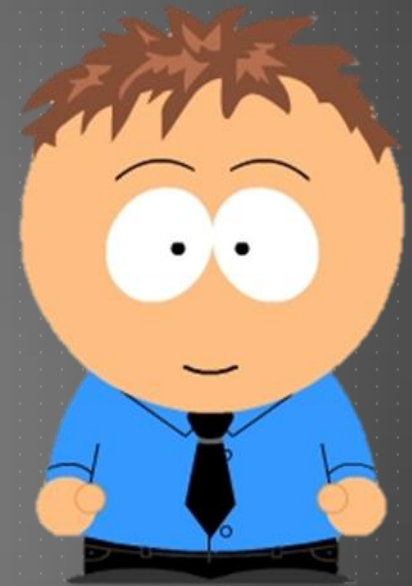
1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(\underbrace{x^2 - 35}_u)$$

$$f'(x) = 2x$$

Differentiate the outer



# SOLUTIONS

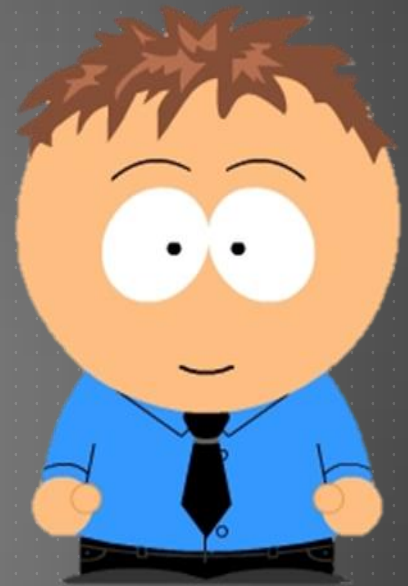
1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(\underbrace{x^2 - 35}_u)$$

$$f'(x) = 2x \cdot \frac{1}{x^2 - 35}$$

Remember the derivative of  
 $\ln(x)$  is  $\frac{1}{x}$



# SOLUTIONS

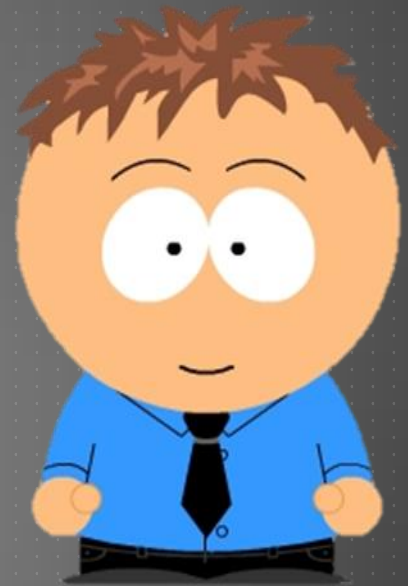
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What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

$$f'(x) = 2x \frac{1}{x^2 - 35}$$

$$f'(x) = \frac{2x}{x^2 - 35}$$



# SOLUTIONS

1.

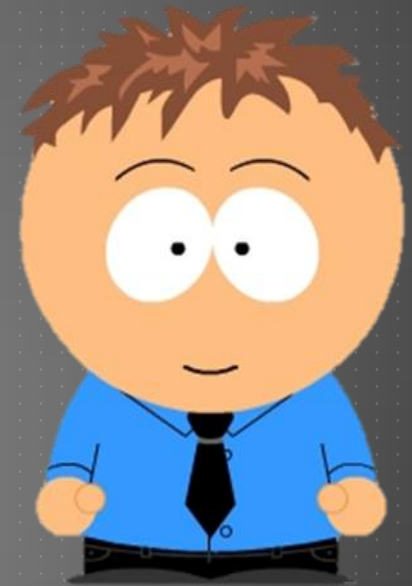
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$$f'(x) = \frac{2x}{x^2 - 35}$$

We want to know when the gradient (derivative) is 1



# SOLUTIONS

1.

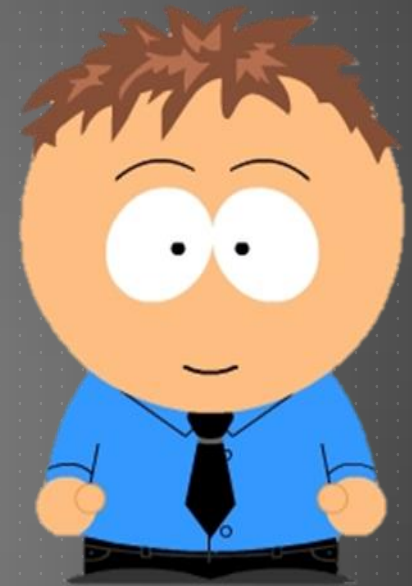
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$$f'(x) = \frac{2x}{x^2 - 35} \quad \frac{2x}{x^2 - 35} = 1$$

We want to know when the gradient (derivative) is 1



# SOLUTIONS

1.

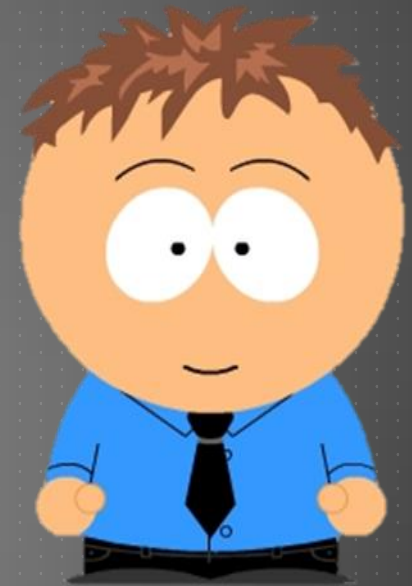
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$$f'(x) = \frac{2x}{x^2 - 35} \quad \frac{2x}{x^2 - 35} = 1$$

Rearrange and solve



# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

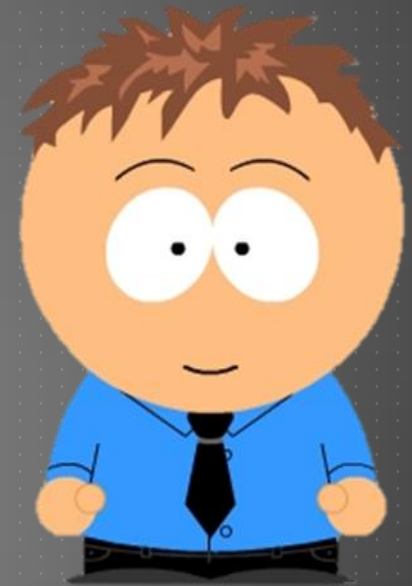
$$f(x) = \ln(x^2 - 35)$$

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$$f'(x) = \frac{2x}{x^2 - 35} \qquad \frac{2x}{x^2 - 35} = 1$$

$$2x = x^2 - 35$$

Rearrange and solve



# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

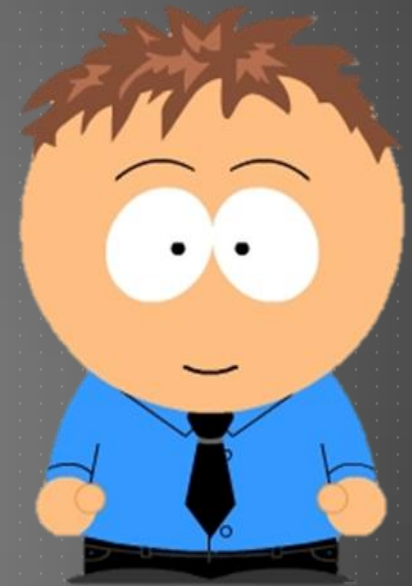
$$f'(x) = 2x \frac{1}{x^2 - 35}$$

$$f'(x) = \frac{2x}{x^2 - 35} \qquad \frac{2x}{x^2 - 35} = 1$$

$$2x = x^2 - 35$$

$$0 = x^2 - 2x - 35$$

Rearrange and solve



# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

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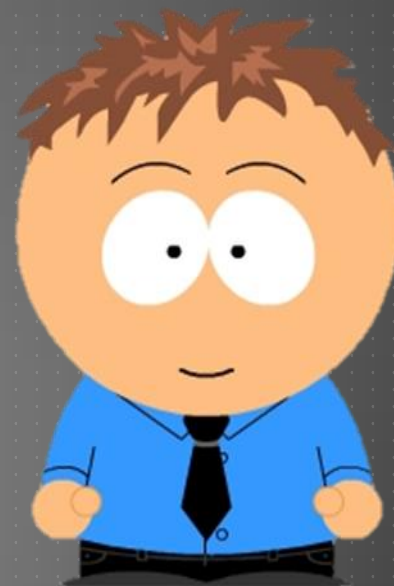
$$f'(x) = \frac{2x}{x^2 - 35} \qquad \frac{2x}{x^2 - 35} = 1$$

$$2x = x^2 - 35$$

$$0 = x^2 - 2x - 35$$

$$x = 7 \text{ or } x = -5$$

Rearrange and solve



# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

$$f'(x) = 2x \frac{1}{x^2 - 35}$$

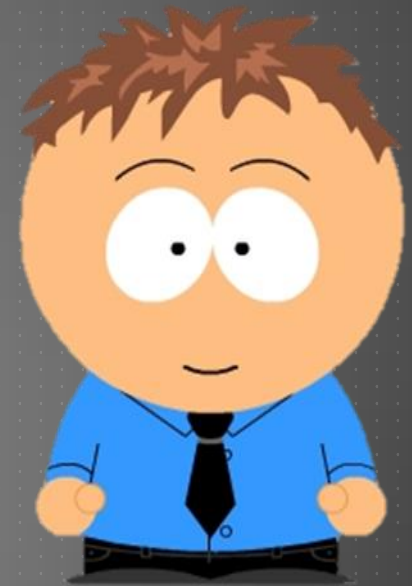
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$$2x = x^2 - 35$$

$$0 = x^2 - 2x - 35$$

$$x = 7 \text{ or } x = -5$$

We have 2 solutions, but we must remember to be careful when dealing with  $\ln$  graphs



# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

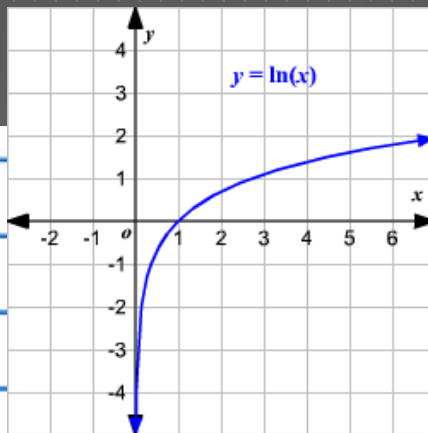
$$f'(x) = 2x \frac{1}{x^2 - 35}$$

$$f'(x) = \frac{2x}{x^2 - 35} \quad \frac{2x}{x^2 - 35} = 1$$

$$2x = x^2 - 35$$

$$0 = x^2 - 2x - 35$$

$$x = 7 \text{ or } x = -5$$



$$\ln(x^2 - 35)$$

Only exists if this part is  $> 0$



# SOLUTIONS

1.

What values of  $x$  does the function  $f(x) = \ln(x^2 - 35)$  have a gradient of 1?

$$f(x) = \ln(x^2 - 35)$$

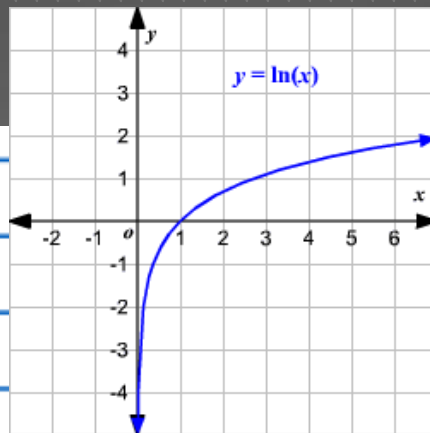
$$f'(x) = 2x \cdot \frac{1}{x^2 - 35}$$

$$f'(x) = \frac{2x}{x^2 - 35} \quad \frac{2x}{x^2 - 35} = 1$$

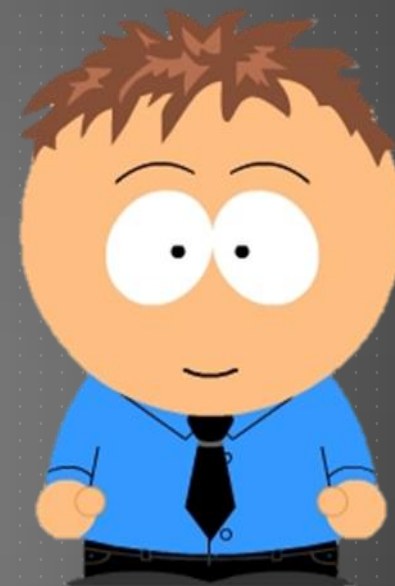
$$2x = x^2 - 35$$

$$0 = x^2 - 2x - 35$$

$$x = 7 \text{ or } x = -5$$



Therefore  $x=7$  is the only solution

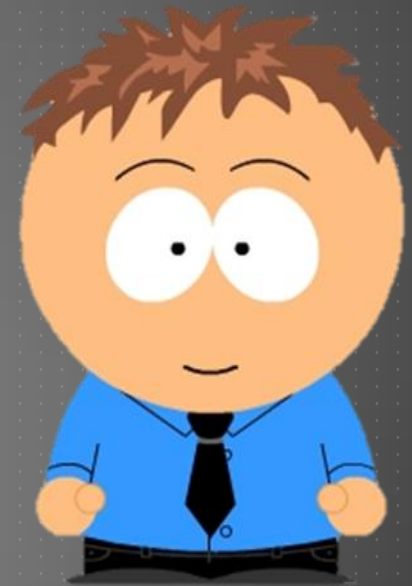


# SOLUTIONS

2.

Find the area under the curve  $y = \frac{1}{2}e^{4x}$ , bounded by the x-axis, and the lines  $y=0$  and  $y=1$

We need to find an area, so that means we must integrate our function



# SOLUTIONS

2.

Find the area under the curve  $y = \frac{1}{2}e^{4x}$ , bounded by the x-axis, and the lines  $y=0$  and  $y=1$

$$\int \frac{1}{2}e^{4x} dx$$

We need to find an area, so that means we must integrate our function

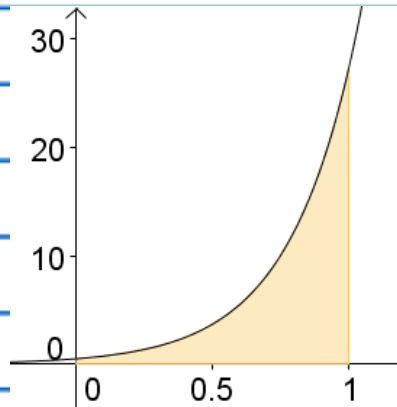


# SOLUTIONS

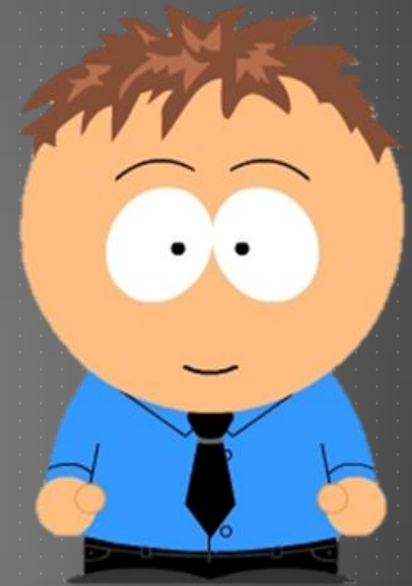
2.

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$$\int \frac{1}{2}e^{4x} dx$$



A quick sketch shows the area we are looking for. There is no section under the x-axis so we can just use limits 0 and 1

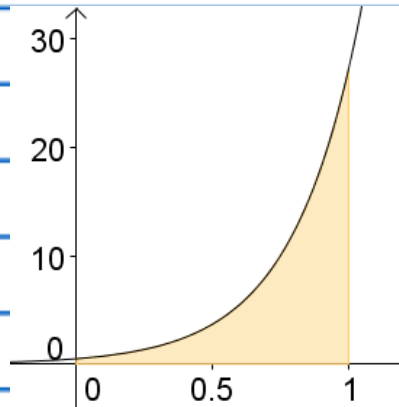


# SOLUTIONS

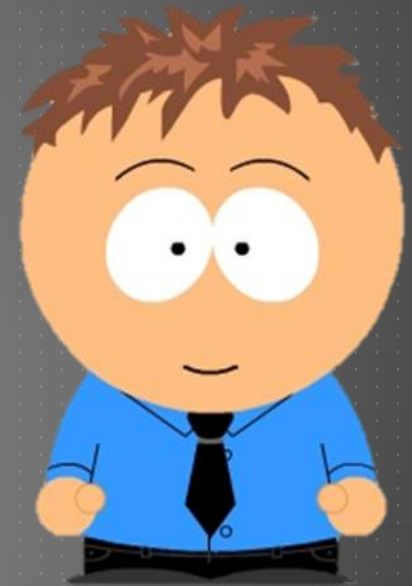
2.

Find the area under the curve  $y = \frac{1}{2}e^{4x}$ , bounded by the x-axis, and the lines  $y=0$  and  $y=1$

$$\int_0^1 \frac{1}{2}e^{4x} dx$$



A quick sketch shows the area we are looking for. There is no section under the x-axis so we can just use limits 0 and 1

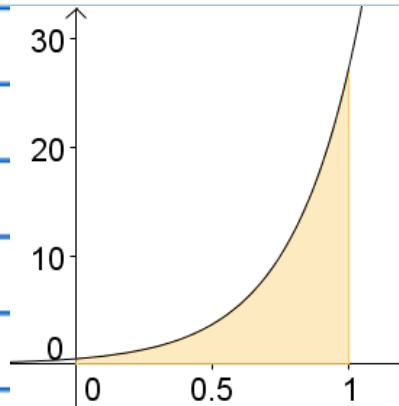


# SOLUTIONS

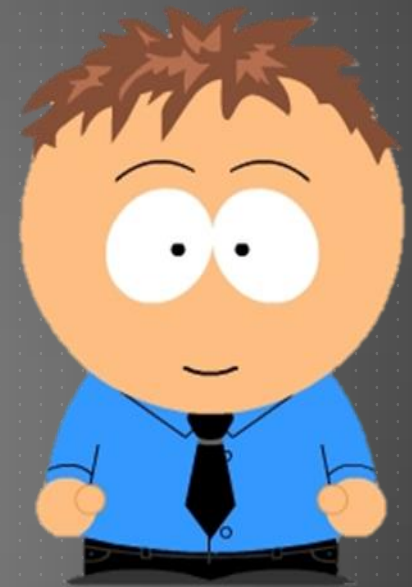
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$$\int_0^1 \frac{1}{2}e^{4x} dx$$



By thinking about what would differentiate to give our answer, quite quickly we can figure out the integral



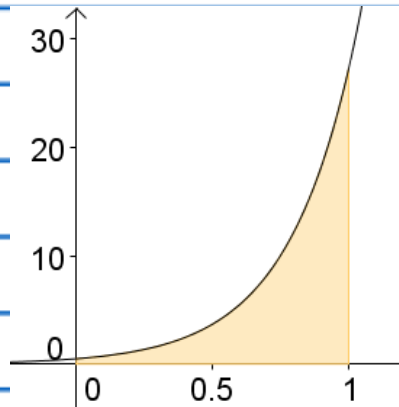
# SOLUTIONS

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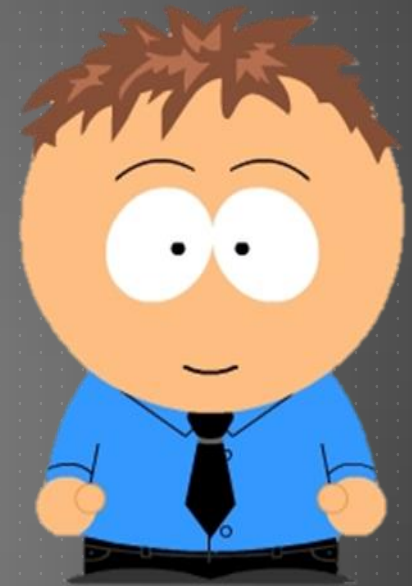
Find the area under the curve  $y = \frac{1}{2}e^{4x}$ , bounded by the x-axis, and the lines  $y=0$  and  $y=1$

$$\int_0^1 \frac{1}{2} e^{4x} dx$$

$$= \left[ \frac{1}{8} e^{4x} \right]_0^1$$



By thinking about what would differentiate to give our answer, quite quickly we can figure out the integral



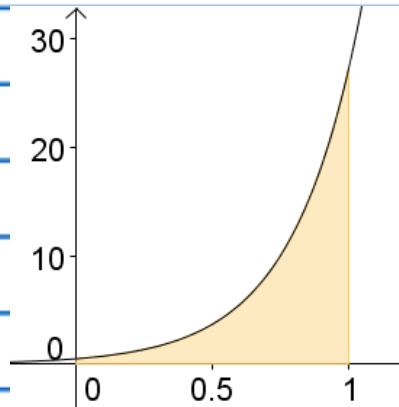
# SOLUTIONS

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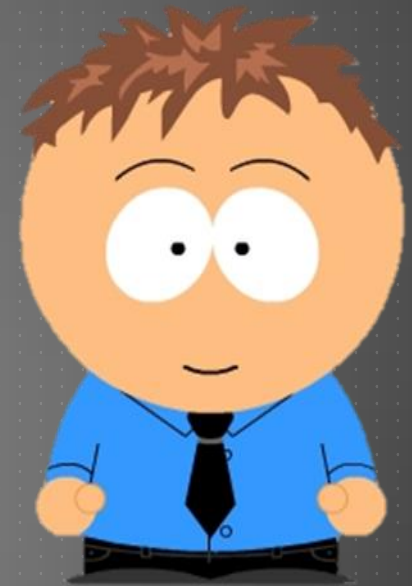
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$$= \left[ \frac{1}{8} e^{4x} \right]_0^1$$



And all that's left to do is evaluate the limits



# SOLUTIONS

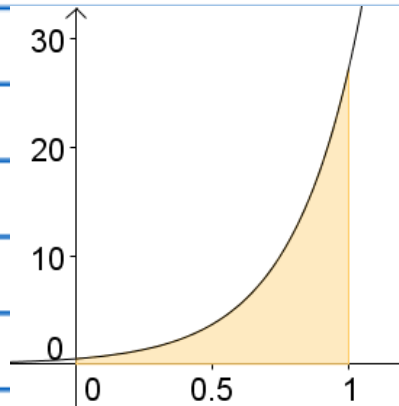
2.

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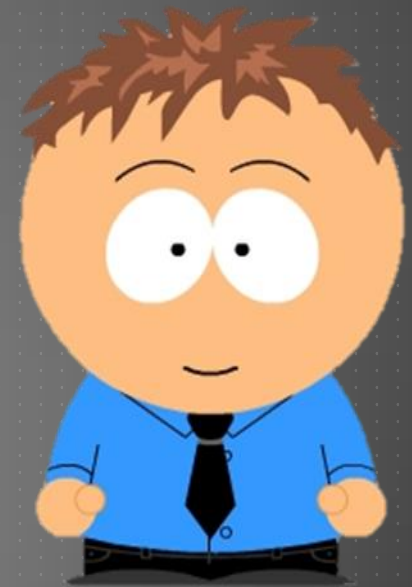
$$\int_0^1 \frac{1}{2} e^{4x} dx$$

$$= \left[ \frac{1}{8} e^{4x} \right]_0^1$$

$$= \left[ \frac{1}{8} e^4 \right] - \left[ \frac{1}{8} e^0 \right]$$



And all that's left to do is evaluate the limits



# SOLUTIONS

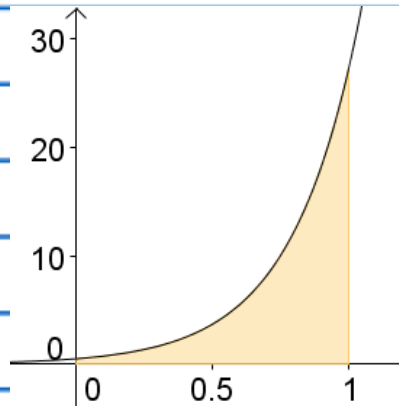
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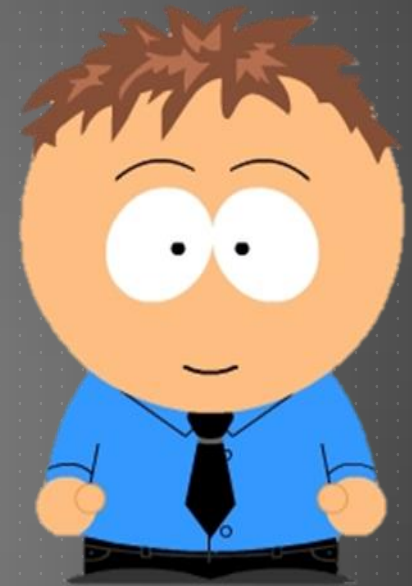
$$\int_0^1 \frac{1}{2} e^{4x} dx$$

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Remembering that  $e^0 = 1$ ...



# SOLUTIONS

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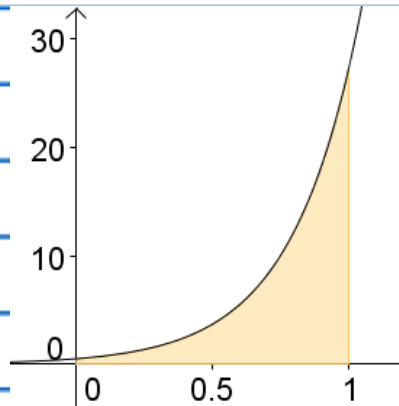
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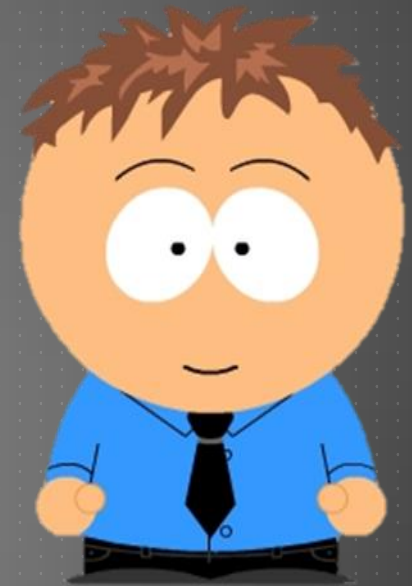
$$= \left[ \frac{1}{8} e^{4x} \right]_0^1$$

$$= \left[ \frac{1}{8} e^4 \right] - \left[ \frac{1}{8} e^0 \right]$$

$$= \frac{1}{8} e^4 - 1$$



Remembering that  $e^0 = 1 \dots$



# SOLUTIONS

2.

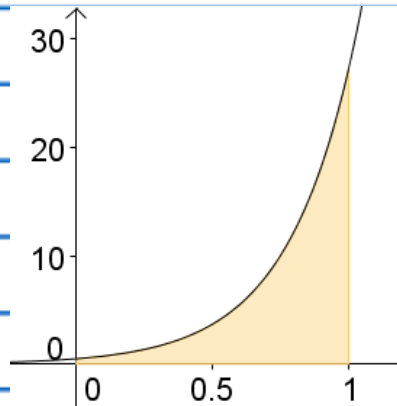
Find the area under the curve  $y = \frac{1}{2}e^{4x}$ , bounded by the x-axis, and the lines  $y=0$  and  $y=1$

$$\int_0^1 \frac{1}{2} e^{4x} dx$$

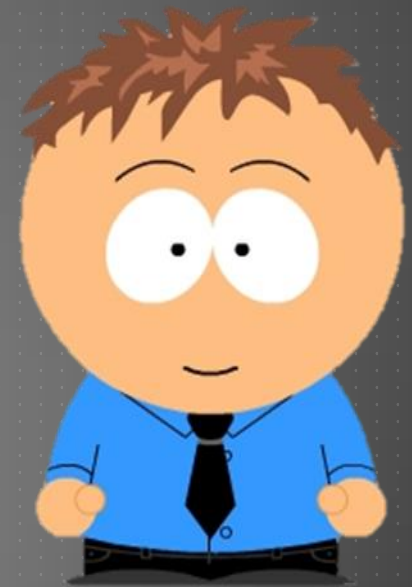
$$= \left[ \frac{1}{8} e^{4x} \right]_0^1$$

$$= \left[ \frac{1}{8} e^4 \right] - \left[ \frac{1}{8} e^0 \right]$$

$$= \frac{1}{8} e^4 - 1$$



You can leave your answer in this form, or evaluate on the GDC



# SOLUTIONS

2.

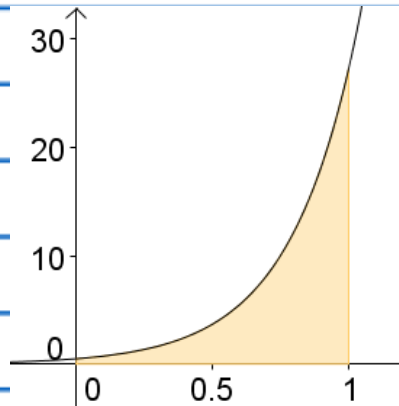
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$$\int_0^1 \frac{1}{2} e^{4x} dx$$

$$= \left[ \frac{1}{8} e^{4x} \right]_0^1$$

$$= \left[ \frac{1}{8} e^4 \right] - \left[ \frac{1}{8} e^0 \right]$$

$$= \frac{1}{8} e^4 - 1 = 5.82 \text{ (3sf)}$$



You can leave your answer in this form, or evaluate on the GDC

