

# TWO BEAUTIFUL PROOFS OF PICK'S THEOREM

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*We present two different proofs of Pick's theorem and analyse in what ways might be perceived as beautiful by working mathematicians. In particular, we discuss two concepts, generality and specificity, that appear to contribute to beauty in different ways. We also discuss possible implications on insight into the nature of beauty in mathematics, and how the teaching of mathematics could be impacted, especially in countries in which discussions of beauty and aesthetics are notably absent from curricular documents.*

**Keywords:** Beauty, aesthetics, proof, Pick's theorem, motivation

## INTRODUCTION

The claim that mathematics contains elements of deep beauty seems uncontroversial. The literature abounds with references to this beauty and descriptions of its nature. For instance S. Chandrasekhar, a Nobel prize winning physicist, once wrote, “a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature, persuades me to say that beauty is that to which the human mind responds at its deepest and most profound” (Chandrasekhar, 1987, p. 54). And G. H. Hardy, a leading mathematician of the 1900's, asserted, “The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way” (Hardy, 1940, p. 85). We even see references to the beauty of mathematics in poetry, such as Edna St. Millay's famous line, “Euclid alone has looked at beauty bare” (Millay, 1941).

What is less clear is why mathematics appears to us as beautiful. Hardy claimed that the sense of beauty comes, at least in part, from a sense of surprise (Hardy, 1940, p. 113). Gian-Carlo Rota refutes this view. He gives Morley's theorem as an example. Morley's theorem states that adjacent angle trisectors of an arbitrary triangle meet in an equilateral triangle. Rota claims that this theorem, while surprising, is not beautiful (Rota, 1997, p. 4) and suggests instead that what characterizes beauty is enlightenment, an admittedly fuzzy concept which he claims mathematicians do their best to avoid (Rota, 1997, p. 13).

Another answer to the question of why mathematics appears as beautiful comes from Elaine Scarry, a professor of English at Harvard, whose work has recently gathered attention even in mathematical circles. In her book “On Beauty and Being Just,” (Scarry, 1998) she claims that beauty is, in essence, compelling. It draws us to itself. This claim resonates with Poincaré, an eminent 19th century mathematician who said, “The scientist does not study nature because it is useful

to do so. He studies it because he takes pleasure in it; and he takes pleasure in it because it is beautiful” (Poincaré, 1952, p. 22). The strong claim is that beauty has some sort of allure, similar to that of pollen to the bee, which draws people’s interest, allows it to replicate, and secures its future.

## **PREVIOUS RESEARCH AND MOTIVATION**

If beauty is compelling, then it seems natural to ask whether it can play a motivational role in the teaching of mathematics. However, there has been surprisingly little research about this question, and related questions, in the area of mathematics education (ESM, 2002). Most of the work done on this area falls under the broader, related notion of aesthetics, which involves a host of affective components in addition to the experience of beauty, such as the experience of pleasure. One recent example comes from Sinclair (2002) who created a model to describe three important functions that aesthetics play in the working lives of mathematicians: generative, motivational, and evaluative<sup>1</sup>. Burton (2004) used this framework to study the aesthetic judgements of mathematicians, looking at the connections between affective experiences of mathematics and intuitions and/or insight that the pursuit of mathematics often provides. Other examples, both from philosophy and mathematics education, include Mack (2006), Tymoczko (1993), and Wang (2001).

This study differs from the previously mentioned in that it specifically investigates the notion of beauty — we are not concerned in this paper with the affective responses, or in fact any other psychological processes related to the aesthetic experience. Another feature of the current study that differs from those previously mentioned is that the main source of data is the mathematics itself, though we draw on pilot data from an interview with one mathematician to help support our analysis, and in a future study will look more closely at interview data. We do not depart from a particular theoretical position, but rather hope to build such a position through a systematic examination of the mathematics, as demonstrated in the analysis below.

A brief look at how beauty is treated in curricular documents in different countries provides some motivation for the eventual outcomes of this project<sup>2</sup>. In some countries, such as China, the aesthetic nature of mathematics is actively researched and explicitly mentioned in the curriculum (e.g. Fu, 2004; Li, 2003; Ministry of China, 2008)<sup>3</sup>. However, in other countries, such as the United States, Sweden, and Finland, there is little or no mention of beauty. In Sweden, there is some explicit mention of beauty at the compulsory level, but not at the high school level (Skolverket, 2000). Moreover, little or no information is given about how beauty should appear in details such as task selection or teaching practices. In the United States, beauty is not listed among the five content and process strands that affect all K-12 levels (NCTM, 2000). Similarly, we

found no mention of beauty in the Finnish curriculum (which does stress many affective qualities, such as “courage” in solving problems)<sup>4</sup>. The kind of work done here could eventually serve a purpose both in raising awareness of beauty in countries that do not currently emphasize it and in articulating some of the features of beauty that could be operationalized in curricular documents.

## CURRENT PROJECT

The research described here is in its initial stages. The ultimate goal, similar to that described in Burton (2004), is to develop a theoretical model for beauty against which we can compare the views held by working mathematicians. To begin creating this model, we have proceeded fairly simple-mindedly. We looked through the literature to find proofs that are commonly held to be beautiful (e.g. Aigner et al., 2010; Wells, 1990). We wanted to choose theorems that would be fairly uncontroversial but not overly discussed (such as a proof of  $e^{\pi i} = -1$ ). We chose to focus on the beauty found in the proofs, not in the theorems themselves (though the two are often linked). We also asked mathematicians to suggest proofs that they consider beautiful, and now have a small collection of these.

One theorem that appeared both in our literature search and as a suggestion from a mathematician was Pick’s theorem, which provides a simple formula for finding the area of a lattice polygon. This theorem is simple enough to be understood and verified by middle schools students, while the statement and proofs of the theorem have relevance for research mathematicians<sup>5</sup>. This feature makes the theorem particularly useful for our study, since on the one hand it is challenging enough to present to mathematicians to obtain data, but on the other hand it is accessible enough to allow us to investigate whether school age children appreciate beauty (or have capacity to appreciate beauty) in similar ways.

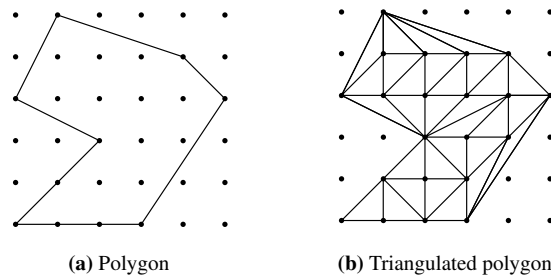
Below we will examine two proofs of the theorem, the first suggested by a mathematician in our pilot study, and the second found in Aigner et al., 2010<sup>6</sup>. These two proofs were chosen because they are similar in many respects, except for one which we would like to highlight, namely a respect that potentially gives rise to a sense of beauty. The two features of beauty that we will highlight (among many more that could be chosen, with other proofs, or even considering different parts of these proofs, such as the lemmas that support them) are those of *generality* and *specificity*. We suspect that these are two features that appear in many instances of mathematical beauty and that they will show up as important features when we interview mathematicians about their judgements of these two particular proofs. The goals of this analysis are to: (1) model the way in which we will analyze proofs in our study; (2) show, via pilot data, how mathematicians’ beliefs might be connected to proof analysis; (3) present our preliminary results to get feedback in order to improve the next iteration of our study.

## PICK'S THEOREM

Pick's theorem gives a simple formula for calculating the area of a lattice polygon, which is a polygon constructed on a grid of evenly spaced points. The theorem, first proven by Georg Alexander Pick in 1899, is a classic result of geometry<sup>7</sup>. An *interior (lattice) point* is a point of the lattice that is properly contained in the polygon, and a *boundary (lattice) point* is a point of the lattice that lies on the boundary of the polygon.

**Theorem** (Pick's theorem). *Let  $A$  be the area of a lattice polygon, let  $I$  be the number of interior lattice points, and let  $B$  be the number of boundary lattice points, including vertices. Then  $A = I + B/2 - 1$ .*

For example, in the lattice polygon given in Figure 1a, there are 10 boundary points and 11 interior points, so the area is  $11 + 10/2 - 1 = 15$ . In any particular case, one can of course confirm that this is correct by dissecting the polygon into suitable triangles and rectangles.



**Figure 1: Example of a lattice polygon, with one possible triangulation**

Below we give two different proofs of Pick's theorem, which we claim are beautiful in different ways. Both proofs use a method of double counting based on a triangulation of the polygon (see Figure 1b). In the first proof, we count the angles inside the triangles in two different ways. In the second proof we interpret the figure as a connected plane graph (a connected network drawn in the plane without crossing edges) and count the number of edges, using Euler's formula to relate the numbers of edges, faces, and vertices of the figure. We will draw on the following three lemmas, which we state here without proof, as they will not figure in the analysis and discussion. An *elementary triangle* is a triangle whose vertices are lattice points, and has no further boundary points and no interior points.

**Lemma 0.1.** *Any lattice polygon can be triangulated by elementary triangles.*

**Lemma 0.2.** *The area of any elementary triangle in the lattice  $\mathbb{Z}^2$  is  $\frac{1}{2}$ .*

**Lemma 0.3** (Euler's formula). *Let  $f$  be the number of faces,  $e$  the number of edges and  $v$  the number of vertices in a connected plane graph. Then  $v - e + f = 2$ .*

## PROOF 1: USING ANGLES

*Proof.* We begin by partitioning the polygon  $P$  into  $N$  elementary triangles, which is possible by Lemma 0.1 (see Figure 1b). We now sum up the internal angles of all of these triangles in two different ways. On the one hand, the angle sum of any triangle is  $\pi$ , so the sum of all the angles is  $S = N \cdot \pi$ . On the other hand, at each interior point  $i$ , the angles of the elementary triangles meeting at  $i$  add up to  $2\pi$ . At each boundary point  $b$  that is not a vertex, the angles of the elementary triangles meeting at  $b$  sum to  $\pi$ . At the vertices, the angles do not add up to  $\pi$ , but if we add the interior angles at all the vertices, we get  $k\pi - 2\pi$ , where  $k$  is the number of vertices, since the sum of the exterior angles is  $2\pi$  (see Figure 2). One can argue for this result by noting that walking along the perimeter of the polygon, one completes one full turn, that is  $2\pi$ . Note that some exterior angles contribute a positive term, and others a negative term.

Let  $I$  be the number of interior points and  $B$  be the number of boundary points. In all, the sum of the angles at boundary points is  $B \cdot \pi - 2\pi$ , and the sum of the angles at internal points is  $I \cdot 2\pi$ . Therefore,  $S = I \cdot 2\pi + B \cdot \pi - 2\pi$ . We conclude that  $N \cdot \pi = I \cdot 2\pi + B \cdot \pi - 2\pi$ , so canceling  $\pi$  we get  $N = 2I + B - 2$ . Since by Lemma 0.2 the area of any elementary triangle is  $\frac{1}{2}$ , we have  $A = \frac{1}{2}N$  and thus  $A = I + \frac{1}{2}B - 1$ .  $\square$

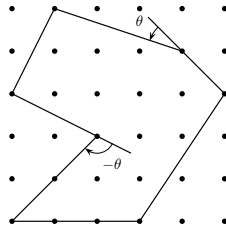


Figure 2: Lattice polygon, with two exterior angles marked

## PROOF 2: USING EULER'S FORMULA

*Proof.* We begin by partitioning  $P$  into elementary triangles, which is possible by Lemma 0.1 (again, see Figure 1b).

We then interpret the triangulation as a connected plane graph, where vertices in the graph are vertices of the triangulation, and edges in the graph are edges of the triangles in the triangulation. This graph subdivides the plane into  $f$  faces, one of which is the unbounded face (the area outside the polygon), and the remaining  $f - 1$  of these are the triangles inside the polygon. By Lemma 0.2, each triangle has area  $\frac{1}{2}$ , and thus  $A = \frac{1}{2}(f - 1)$ . This of course proves nothing; it is a simple consequence of how we defined  $f$ .

An *interior* edge borders on two triangles, and a *boundary* edge borders on a sin-

gle triangle and forms part of the boundary of the polygon itself. Let  $e_{int}$  be the number of interior edges, and  $e_{bd}$  be the number of boundary edges. Counting the number of edges in two different ways, we get

$$3(f - 1) = 2e_{int} + e_{bd}. \quad (*)$$

Here we are overcounting to get the total number of edges of the collection of triangles. The left hand side counts these edges using the fact that each triangle has 3 edges. The right hand side counts them using the fact that each interior edge contributes to two triangles, while each exterior edge contributes to one triangle. We can also observe that the number of boundary edges is the same as the number of boundary vertices,  $B = e_{bd}$ , and that the number of vertices in the graph is the sum of all the interior and boundary points,  $v = I + B$ . Euler's formula for the graph at hand states that

$$(I + B) - e + f = 2, \quad \text{or} \quad e - f = (I + B) - 2$$

where  $e = e_{int} + e_{bd}$  is the total number of edges. With some clever algebraic rearrangements, starting with (\*), we get

$$\begin{aligned} f &= -2f + 3 + 2e_{int} + e_{bd} \\ &= -2f + 3 + 2e - e_{bd} \\ &= 2(e - f) - e_{bd} + 3 \\ &= 2(I + B - 2) - B + 3 = 2I + B - 1. \end{aligned}$$

Thus we get  $A = \frac{1}{2}(f - 1) = \frac{1}{2}((2I + B - 1) - 1) = I + \frac{1}{2}B - 1$ . □

## ANALYSIS AND DISCUSSION

To what extent are each of these proofs beautiful? We begin with some data from a mathematician who thought the first proof was beautiful. One way in which the proof is beautiful to him is that it gives meaning to the terms  $I$ ,  $\frac{B}{2}$ , and  $-1$ . He explains, "In particular, I like that you can see that each boundary lattice point contributes half as much total angle as each interior lattice point." He also said that he likes proofs that get information by counting the same things in different ways. The particular choice of counting angle measures, though, both contributed and detracted from the sense of beauty in this proof. He says, "The fact that the proof involves angles is beautiful in the sense that it is unexpected, but also ugly in that it breaks some symmetry." Pick's theorem, as stated, holds for any lattice polygons, regardless of whether the lattice itself is transformed in a way that preserves area. For example, if you shear the triangle, the area is unchanged, but the angle measures change<sup>8</sup>. Thus the introduced new quantity, angles, does not have the same property as the figure itself, which this mathematician referred to as "unnatural".

In contrast, the second proof, using Euler's formula, uses only quantities that are invariant under transformation. What seems beautiful about it is that it turns out to be an application of Euler's formula. One gets a sense of "even here, this method can apply!" But whereas the second proof is more general than the first (we introduce no auxiliary concepts) it is much less intuitive. The first proof, besides the sophisticated application of double counting, is fairly elementary. Even a grade school child can count the angle measures in both ways described above. However, the second proof requires a bit more machinery to understand. First you must conceptualize the plane in such a way that Euler's formula applies (which includes the somewhat strange step of considering the complement of the polygon in the plane as a face in itself.) Also, in applying Euler's formula, one is resting on a result which by itself is not obvious. Even if one really believes Euler's formula and feels comfortable using it to get the result, one doesn't get a full understanding of the proof if one doesn't in turn understand why Euler's formula is true. This reliance on heavy theory seems to be an aspect which detracts from the beauty of the second proof.

We see then that in each of the proofs there is some feature that contributes to the beauty and some feature that detracts. It turns out in this case that the features are complementary. The feature that contributes to the beauty of the first proof is missing in the second and vice versa. For instance, in the second proof, what makes it beautiful is some sort of generality. This particular proof fits into a family of proofs all of which are instances of Euler's formula. In the first proof, what makes it beautiful is some sort of specificity. The surprising use of angle measures in the double counting introduces some unexpected element, which on the one hand breaks the harmony of the proof, but on the other hand — perhaps because of that breaking — becomes a compelling feature of the argument.

Both proofs appear to contain an element of surprise, but the nature of that surprise is almost opposite. In the first case the surprise arises from the specificity. We contend that the pleasure one gets from reading the proof is similar to the feeling of finding a specific tool, like the correct size hexagonal screwdriver for one particular screw. In the second case the surprise comes from the feeling of generality. The sense of fitting in arises from there being a set of objects that have a similar property. It is a wonderful, unexpected finding that this second proof is one of those kinds of proofs. To continue the tool analogy, Euler's formula is the monkey wrench, that is suitable for a great number of different situations.

## CONCLUDING COMMENTS AND NEXT STEPS

To claim that generality and specificity contribute to the beauty of these proofs through some element of surprise, we must return to Rota's criticism that beauty arises out a feeling of enlightenment rather than surprise. His critique was

grounded on the fact that there are proofs that are surprising, but nonetheless not beautiful. For now we leave this as an open question, with the possibility that surprise might be a necessary (or at least contributing) but not sufficient condition for beauty. We note, however, that what seems similar about surprise and enlightenment is some sort of allure, something that grabs the mind's interest. And it might be this allure, or "compelling"-ness referring back to Scarry again, that is the defining characteristic of beauty.

It seems fairly obvious to say that for a proof to be compelling, it must on the one hand be not too simple, and on the other hand not too complex for the mind to grasp. It might be that the features of generality and specificity are what keeps these two particular proofs appropriately compelling. The specificity of the first one makes the proof technically accessible. The generality of the second one imparts a certain status. Another striking characteristic of these two features is the fact that they play mirrored roles with each other — they are in a sense duals — and the way in which they mirror each other is that one contributes to beauty in exactly the way that the other detracts. At first the fact that two seemingly opposite characteristics could both give rise to beauty might seem contradictory, but we offer another interpretation: that beauty arises from the interplay of a sort of access and restraint. A potentially beautiful object which was completely accessible might not appear beautiful, just as a potentially beautiful object that is completely hidden would never be able to be experienced as beautiful. The proof based on Euler's theorem brings out some sort of hidden structure; the proof based on angle measures provides a specific instantiation of an otherwise seemingly common sort of mathematical tool. Generality and specificity might not be just two features of beauty— they might turn out to be *exactly* the aspects of mathematical expression which provide the needed tension to give rise to the sense of beauty. We do not rule out that there might be other features that give rise to beauty, but from our preliminary analysis, we are willing to commit that the fact that we found these two particular features here is not surprising, nor idiosyncratic.

This study was meant as a first step into a rather large inquiry domain. The goal was to make the pursuit of a study of beauty in mathematics tractable, both in terms of methods and potential results. This study gives rise to a few hypotheses that we would like to investigate (and invite others to investigate!) in future studies. These include:

- The features of generality and specificity are not idiosyncratic. They appear in a wide number of proofs commonly held to be beautiful.
- There is consensus among mathematicians, not just about which proofs and/or theorems are beautiful, but also about what gives rise to the sense of beauty.



- The fact that generality and specificity are related, as duals, is also not a coincidence. If there are other features that give rise to beauty, they will also be related in a way that creates some sort of tension, and the sense of beauty that arises will be related to this tension.

## NOTES

1. A description of these functions is given in Sinclair (2002): “The most recognized and public of the three roles of the aesthetic is the evaluative; it concerns the aesthetic nature of mathematical entities and is involved in judgments about the beauty, elegance, and significance of entities such as proofs and theorems. The generative role of the aesthetic is a guiding one and involves nonpropositional modes of reasoning used in the process of inquiry. I use the term generative because it is described as being responsible for generating new ideas and insights that could not be derived by logical steps alone. Lastly, the motivational role refers to the aesthetic responses that attract mathematicians to certain problems and even to certain fields of mathematics.”
2. Thanks to the following people who provided information on statements about beauty and aesthetics in curricular documents: Antti Viholainen (Finland), Kirsti Hemmi (Sweden), Aihui Peng (China). We welcome examples from other countries, especially those that incorporate beauty in a meaningful way.
3. Some examples of curricular statements include “appreciate the aesthetic value of mathematics theorems and mathematics methods”, “experience the flexibility, the elegance (similar to the beauty, but higher than beauty, and ingenuity of mathematical proof”, “experience the beauty of figure”.
4. See [http://www.oph.fi/english/publications/2009/national\\_core\\_curricula\\_for\\_basic\\_education](http://www.oph.fi/english/publications/2009/national_core_curricula_for_basic_education).
5. See [http://www.cut-the-knot.org/ctk/Pick\\_proof.shtml](http://www.cut-the-knot.org/ctk/Pick_proof.shtml) for a web application of a classic way of introducing the task to middle school students. And see Sally & Sally (2007) for a lovely exposition of how this task can be made relevant to people of all ages, from school children to research mathematicians, not just in terms of verifying the theorem, but in terms of really understanding the underlying ideas.
6. These two proofs correspond closely to the two proofs given in Sally & Sally (2007).
7. The original proof is found in Pick (1899), and some history given at <http://jsoles.myweb.uga.edu/history.html>.
8. However, the *argument* involving angle measures (Proof 1) works for transformed lattice polygons as well.

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